Naive Bayes Classification

Professor Ameet Talwalkar

Outline

Registering for Course

- No more issues – there is now space and you can register online

Homework 2

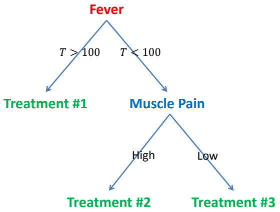
HW1

- Available online since last Friday
- Due next Wednesday (2/1)

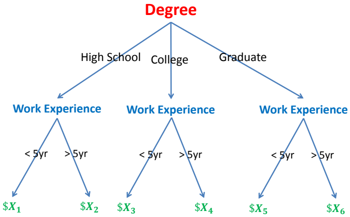
Outline

Many decisions are tree structures

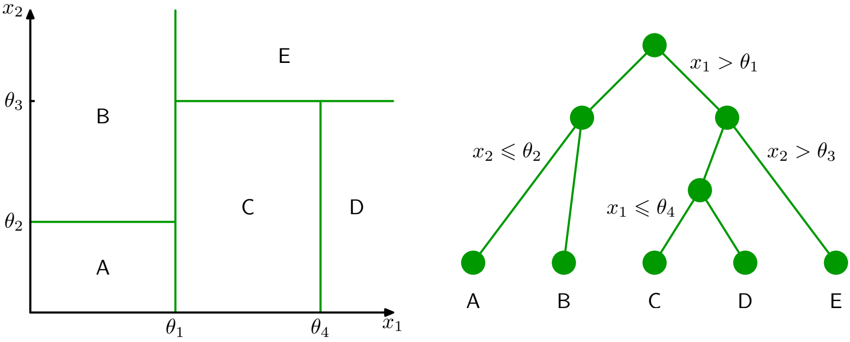
Medical treatment



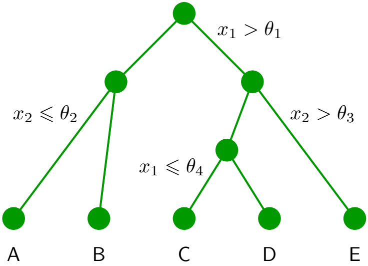
Salary in a company



A tree partitions the feature space

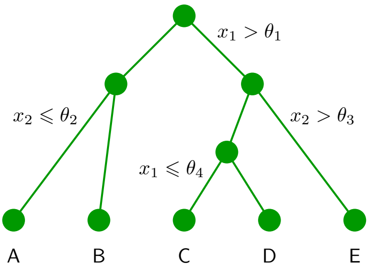


Learning a tree model



Three things to learn:

Learning a tree model

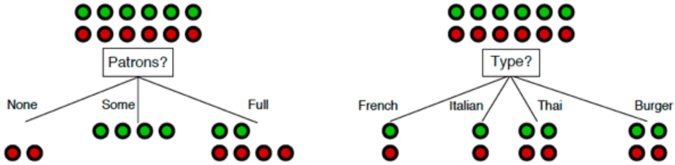


Three things to learn:

1. The structure of the tree.
2. The threshold values (θ_i).
3. The values for the leafs ($A, B, \ldots$).

First decision: at the root of the tree

Which attribute to split?



Patrons? is a better choice—gives **information** about the classification

Idea: use information gain to choose
which attribute to split

How to measure information gain?

Idea:

Gaining information reduces uncertainty

the base can be 2 , though it is not essential (if the base is 2, the unit of the entropy is called "bit")

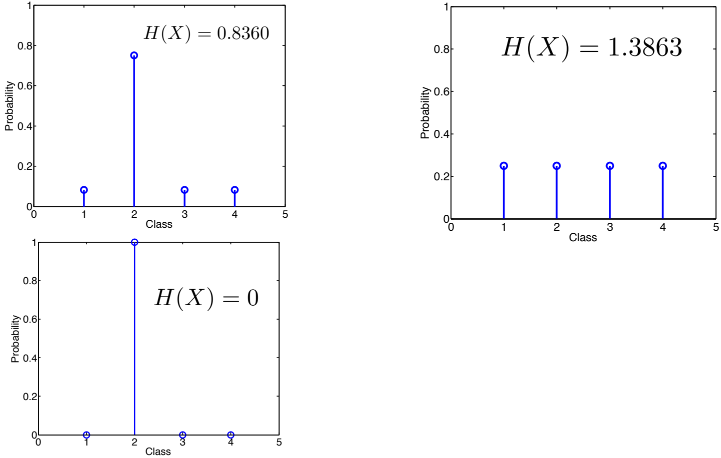
Use to entropy to measure uncertainty

If a random variable X has K different values, a_1, a_2, ...a_K, it is entropy is given by

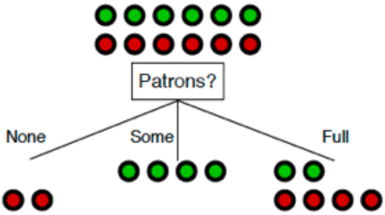
$$H[X] = -\sum_{k=1}^{K} P(X = a_k) \log P(X = a_k)$$

Examples of computing entropy

Entropy



Do we split on “Non” or “Some”?



No, we do not

The decision is deterministic, as seen from the training data

What is the optimal Tree Depth?

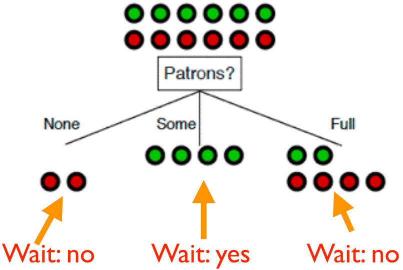
What is the optimal Tree Depth?

- We need to be careful to pick an appropriate tree depth
  - If the tree is too deep, we can overfit
  - If the tree is too shallow, we underfit
- Max depth is a hyperparameter that should be tuned by the data
- Alternative strategy is to create a very deep tree, and then to prune it (see Section 9.2.2 in ESL for details)
- If leaves aren't completely pure, we predict using majority vote

Example

Example

We stop after the root (first node)



Computational Considerations

Numerical Features

Computational Considerations

Numerical Features

- We could split on any feature, with any threshold
- However, for a given feature, the only split points we need to consider are the the n values in the training data for this feature.
- If we sort each feature by these n values, we can quickly compute our impurity metric of interest (cross entropy or others)
  - This takes $O(dn \log n)$ time

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Categorical Features

- Assuming q distinct categories, there are $2^{q-1} - 1$ possible partitions we can consider.
- However, things simplify in the case of binary classification (or regression), and we can find the optimial split (for cross entropy and Gini) by only considering $q - 1$ possible splits (see Section 9.2.4 in ESL for details).

Summary of learning trees

Advantages of using trees

- Can be interpreted by humans
- Computationally efficient
- Handles both numerical and categorical data
- It is parametric thus compact: unlike NNC, we do not have to carry our training instances around
- Building block for various ensemble methods (more on this later)

Disadvantages

- Heuristic training techniques
  - Finding partition of space that minimizes empirical error is NP-hard
  - We resort to greedy approaches with limited theoretical underpinnings

Outline

1. Administration
2. Review of last lecture
3. Naive Bayes
   - Motivating Example
   - Naive Bayes Model
   - Parameter Estimation

A daily battle

I'm going to be rich!!

FROM THE DESK OF MR. AMINU SALEH
DIRECTOR, FOREIGN OPERATIONS DEPARTMENT
AFRI BANK PLC
Afribank Plaza,
14th Floormoney344.jpg
51/55 Broad Street,
P.M.B 12021 Lagos-Nigeria

Attention: Honorable Beneficiary,

IMMEDIATE PAYMENT NOTIFICATION VALUED AT **US$10 MILLION**

It is my modest obligation to write you this letter in regards to the authorization of your owed payment through our most respected financial institution (AFRI BANK PLC). I am Mr. Aminu Saleh, The Director, Foreign Operations Department, AFRI Bank Plc, NIGERIA. The British Government, in conjunction with the US GOVERNMENT, WORLD BANK, UNITED NATIONS ORGANIZATION on foreign payment matters, has empowered my bank after much consultation and consideration, to handle all foreign payments and release them to their appropriate beneficiaries with the help of a representative from Federal Reserve Bank.

To facilitate the process of this transaction, please kindly re-confirm the following information below:

1) Your full Name and Address:
2) Phones, Fax and Mobile No. :
3) Profession, Age and Marital Status:
4) Copy of any valid form of your Identification:

How to tell spam from ham?

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51/55 Broad Street,
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Attention: Honorable Beneficiary,

IMMEDIATE PAYMENT NOTIFICATION VALUED AT **US$10 MILLION**

Dear Ameet,

Do you have 10 minutes to get on a videocall before 2pm?

Thanks,

Stefano

How might we create features?

Intuition

Q: How might a human solve this problem?

A: Simple strategy would be to look for keywords that we often associate with spam

Spam emails

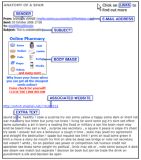
we expect to see words like "money", "free", "bank account", "viagra"

Ham emails

word usage is more spread out with few 'spammy' words

Simple strategy: count the words

Bag-of-word representation of documents (and textual data)



$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

Just wanted to send a quick reminder about the guest lect
noon. We meet in RTH 105. It has a PC and LCD projector
connection for your laptop if you desire. Maybe we can
to setup the A/V stuff.

Again, if you would be able to make it around 30 minutes
great.

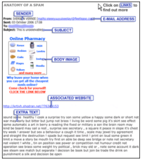
Thanks so much for your willingness to do this,
Mark

$$\begin{pmatrix} \text{free} & 1 \\ \text{money} & 1 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

Weighted sum of those telltale words

different weights for spam and ham: representing how compatible the word pattern is to each category

$$\begin{pmatrix} 100 \times 0.2 \\ 2 \times 0.3 \\ \vdots \\ 2 \times 0.3 \\ \vdots \end{pmatrix}$$



$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.01 \\ 2 \times 0.02 \\ \vdots \\ 2 \times 0.01 \\ \vdots \end{pmatrix}$$

Weighted sum of those telltale words

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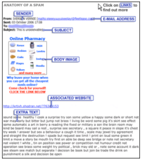


$$\begin{pmatrix} \text{free} & 100 \\ \text{money} & 2 \\ \vdots & \vdots \\ \text{account} & 2 \\ \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 100 \times 0.2 \\ 2 \times 0.3 \\ \vdots \\ 2 \times 0.3 \\ \vdots \end{pmatrix}$$

= 3.2

$$\begin{pmatrix} 100 \times 0.01 \\ 2 \times 0.02 \\ \vdots \\ 2 \times 0.01 \\ \vdots \end{pmatrix}$$

= 1.03

Our intuitive model of classification

Assign weight to each word

Compute compatibility score to "spam"

of "free" x a_{free} + # of "account" x $a_{account}$ + # of "money" x a_{money}

Compute compatibility score to "ham":

of "free" x b_{free} + # of "account" x $b_{account}$ + # of "money" x b_{money}

Make a decision:

if spam score > ham score then spam

else ham

How do we get the weights?

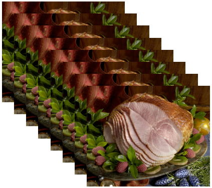
How do we get the weights?

Learn from experience

get a lot of spams

get a lot of hams

But what to optimize?



A probabilistic modeling perspective

Naive Bayes model for identifying spam

Class label: binary

y = {spam, ham}

Features: word counts in the document (Bag-of-word)

Ex: x = {('free', 100), ('lottery', 10), ('money', 10), , ('identification', 1)...}

Each pair is in the format of (w_i, #w_i), namely, a unique word in the dictionary, and the number of times it shows up

Naive Bayes Model (Intuitively)

Features: word counts in the document

Ex: x = {('free', 100), ('identification', 2), ('lottery', 10), ('money', 10), ...}

Model: Naive Bayes (NB)

$$\begin{aligned} p(x|\text{spam}) = p(\text{'free'}|\text{spam})^{100} p(\text{'identification'}|\text{spam})^{2} \\ p(\text{'lottery'}|\text{spam})^{10} p(\text{'money'}|\text{spam})^{10} \cdots \end{aligned}$$

Naive Bayes Model (Intuitively)

Features: word counts in the document

Ex: x = {('free', 100), ('identification', 2), ('lottery', 10), ('money', 10), ...}

Model: Naive Bayes (NB)

$$p(x|\text{spam}) = p(\text{'free'}|\text{spam})^{100} p(\text{'identification'}|\text{spam})^{2}$$
$$p(\text{'lottery'}|\text{spam})^{10} p(\text{'money'}|\text{spam})^{10} \cdots$$

Parameters to be estimated are conditional probabilities: p('free'|spam), p('free'|ham),etc

Naive Bayes Model

- Intuitively this makes some sense (even if it seems simple)
- We'll now discuss the following:
  - Formal modeling assumptions for NB, and why it's 'naive'
  - NB classification rule converges to Bayes Optimal under these assumptions
  - How to estimate model parameters

Naive Bayes Model

$$
\begin{aligned}
p(x|y) &= p(w_1|y)^{\#w_1} p(w_2|y)^{\#w_2} \cdots p(w_m|y)^{\#w_m} \\
&= \prod_i p(w_i|y)^{\#w_i}
\end{aligned}
$$

These conditional probabilities are model parameters

Recall that each data point is a tuple (w_i, $\#w_i$), namely, a unique dictionary word and the # of times it shows up

What is naive about this?

Why is this 'Naive'

Strong assumption of conditional independence:

$$p(w_i, w_j | y) = p(w_i | y) p(w_j | y)$$

Previous example: $p(x|\text{spam}) = p(\text{'free'}|\text{spam})^{100} p(\text{'identification'}|\text{spam})^{2}$
$p(\text{'lottery'}|\text{spam})^{10} p(\text{'money'}|\text{spam})^{10} \cdots$

This assumption makes estimation much easier (as we'll see)

Naive Bayes classification rule

For any document x, we want to compare

$$p(\text{spam}|x) \quad \text{and} \quad p(\text{ham}|x)$$

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Recall that Bayes Optimal classifier uses the posterior probability

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Recall that Bayes Optimal classifier uses the posterior probability

$$f^*(\boldsymbol{x}) = \begin{cases} 1 & \text{if } p(y=1|\boldsymbol{x}) \geq p(y=0|\boldsymbol{x}) \\ 0 & \text{if } p(y=1|\boldsymbol{x}) < p(y=0|\boldsymbol{x}) \end{cases}$$

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NB classification rule looks like the Bayes Optimal classifier under the assumption of conditional independence we just described

Naive Bayes classification rule

For any document x, we want to compare $p(\text{spam}|x)$ and $p(\text{ham}|x)$

Axiom of Probability: $p(\text{spam}, x) = p(\text{spam}|x)p(x) = p(x|\text{spam})p(\text{spam})$

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This gives us (via bayes rule):

$$p(\text{spam}|x) = \frac{p(x|\text{spam})p(\text{spam})}{p(x)}$$

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$$p(\text{spam}|x) = \frac{p(x|\text{spam})p(\text{spam})}{p(x)}$$

$$p(\text{ham}|x) = \frac{p(x|\text{ham})p(\text{ham})}{p(x)}$$

Denominators are same, and easier to compute logarithms, so we compare:

$$\log[p(x|\text{spam})p(\text{spam})] \quad \text{versus} \quad \log[p(x|\text{ham})p(\text{ham})]$$

Classifier in linear form

$$\log[p(x|\mathsf{spam})p(\mathsf{spam})] = \log\left[\prod_i p(w_i|\mathsf{spam})^{\#w_i} p(\mathsf{spam})\right] \quad (1)$$

$$= \sum_i \#w_i \log p(w_i|\mathsf{spam}) + \log p(\mathsf{spam}) \quad (2)$$

Classifier in linear form

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Similarly, we have

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Classifier in linear form

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$$\log[p(x|\mathsf{ham})p(\mathsf{ham})] = \sum_i \#w_i \log p(w_i|\mathsf{ham}) + \log p(\mathsf{ham})$$

We're back to the idea of comparing weighted sums of word occurrences!

$\log p(\mathit{spam})$ *and* $\log p(\mathit{ham})$ *are called "priors" (in our initial example we did not include them but they are important!)*

Mini-summary

What we have shown

By assuming a probabilistic model (i.e., Naive Bayes), we are able to derive a decision rule that is consistent with our intuition

Our next step is to learn the parameters from data

What are the parameters to learn?

Formal definition of Naive Bayes

General case

Given a random variable $X \in \mathbb{R}^{\mathsf{D}}$ and a dependent variable $Y \in [\mathsf{C}]$, the Naive Bayes model defines the joint distribution

$$P(X = x, Y = c) = P(Y = c)P(X = x|Y = c) \tag{3}$$

$$= P(Y = c) \prod_{d=1}^{\mathsf{D}} P(X_d = x_d|Y = c) \tag{4}$$

Special case (i.e., our model of spam emails)

Assumptions

- All X_d are categorical variables from the same domain — $x_d \in [\mathsf{K}]$, for example, the index to the unique words in a dictionary.
- $P(X_d = x_d | Y = c)$ depends only on the value of x_d, not d itself, namely, order is not important (we only care about counts).

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Simplified definition

$$P(X = x, Y = c) = P(Y = c) \prod_k P(k|Y = c)^{z_k} = \pi_c \prod_k \theta_{ck}^{z_k}$$

where z_k is the number of times the kth word appears in x.

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Since order doesn't matter, we enumerate over the cardinality of the domain, rather than the over the number of features D (as in last slide).

Learning problem

Training data

$$\mathcal{D} = \{(x_n, y_n)\}_{n=1}^{\mathsf{N}} \rightarrow \mathcal{D} = \{(\{z_{nk}\}_{k=1}^{\mathsf{K}}, y_n)\}_{n=1}^{\mathsf{N}}$$

Goal

Learning problem

Training data

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Goal

Learn $\pi_c, c = 1, 2, \cdots, \mathsf{C}$, and $\theta_{ck}, \forall c \in [\mathsf{C}], k \in [\mathsf{K}]$ under the constraints

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Goal

Learn $\pi_c, c = 1, 2, \cdots, \mathsf{C}$, and $\theta_{ck}, \forall c \in [\mathsf{C}], k \in [\mathsf{K}]$ under the constraints

$$\sum_c \pi_c = 1$$

and

$$\sum_k \theta_{ck} = \sum_k P(k|Y = c) = 1$$

as well as those quantities should be nonnegative.

Our hammer: maximum likelihood estimation

Recall our joint probability

$$P(X = x, Y = c) = \pi_c \prod_k \theta_{ck}^{z_k}$$

where z_k is the number of times the kth word appears in x.

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Likelihood of the training data

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$$L = P(\mathcal{D}) = \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(x_n | y_n)$$

Our hammer: maximum likelihood estimation

Log-Likelihood of the training data

$$\mathcal{L} = \log P(\mathcal{D}) = \log \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(x_n | y_n)$$

Our hammer: maximum likelihood estimation

Log-Likelihood of the training data

$$\mathcal{L} = \log P(\mathcal{D}) = \log \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(x_n | y_n)$$

$$= \log \prod_{n=1}^{\mathsf{N}} \left(\pi_{y_n} \prod_k \theta_{y_n k}^{z_{nk}} \right)$$

Our hammer: maximum likelihood estimation

Log-Likelihood of the training data

$$\begin{aligned}
\mathcal{L} = \log P(\mathcal{D}) &= \log \prod_{n=1}^{\mathsf{N}} \pi_{y_n} P(x_n | y_n) \\
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&= \sum_n \left(\log \pi_{y_n} + \sum_k z_{nk} \log \theta_{y_n k} \right)
\end{aligned}$$

Our hammer: maximum likelihood estimation

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$$
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\end{aligned}
$$

Our hammer: maximum likelihood estimation

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&= \sum_n \log \pi_{y_n} + \sum_{n,k} z_{nk} \log \theta_{y_n k}
\end{aligned}$$

Optimize it!

$$(\pi_c^*, \theta_{ck}^*) = \arg\max \sum_n \log \pi_{y_n} + \sum_{n,k} z_{nk} \log \theta_{y_n k}$$

Details

Note the separation of parameters in the likelihood

$$\sum_n \log \pi_{y_n} + \sum_{n,k} z_{nk} \log \theta_{y_n k}$$

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The latter implies $\{\theta_{ck}\}$ and $\{\theta_{c'k}\}$ for $c \neq c'$ can be estimated independently!

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Solution

$$\pi_c^* = \frac{\#\text{of data points labeled as c}}{\text{N}}$$

Estimating $\{\theta_{ck}, k = 1, 2, \cdots, \mathsf{K}\}$

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We want to maximize

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Intuition

- Again similar to roll a dice: each side of the dice shows up with a probability of θ_{ck} (total K sides)
- And we have total $\sum_{n:y_n=c,k} z_{nk}$ trials.

Solution

$$\theta^*_{ck} = \frac{\text{\#of times side k shows up in data points labeled as c}}{\text{\#total trials for data points labeled as c}}$$

Translating back to our problem of detecting spam emails

- Collect a lot of ham and spam emails as training examples
- Estimate the "prior"

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$$p(\text{funny\_word}|\text{ham}) = \frac{\#\text{of funny\_word in ham emails}}{\#\text{of words in ham emails}} \tag{5}$$

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$$y^* = \arg\max_{c \in [\mathsf{C}]} P(y = c|x) \tag{7}$$

$$= \arg\max_{c \in [\mathsf{C}]} P(y = c) P(x|y = c) \tag{8}$$

$$= \arg\max_c [\log \pi_c + \sum_k z_k \log \theta_{ck}] \tag{9}$$

Constrained optimization

Equality Constraints

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0 \end{aligned}$$

Method of Lagrange multipliers

Construct the following function (Lagrangian)

$$L(x, \lambda) = f(x) + \lambda g(x)$$

A short derivation of the maximum likelihood estimation

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We can use the method of Lagrangian multipliers!

$$f(x) = -\sum_{n:y_n=c,k} z_{nk} \log \theta_{ck}$$

$$g(x) = \sum_k \theta_{ck} - 1 = 0$$

Chris will review in section on Friday

Summary

You should know or be able to

- What naive Bayes model is
  - write down the joint distribution
  - explain the conditional independence assumption implied by the model
  - explain how this model can be used to classify spam vs ham emails

- Be able to go through the short derivation for parameter estimation
  - The model illustrated here is called discrete/multinomial Naive Bayes
  - HW2 asks you to apply the same principle to Gaussian naive Bayes
  - The derivation is very similar – except there you need to estimate Gaussian continuous random variables (instead of estimating discrete random variables like rolling a dice)

Moving forward

Examine the classification rule for naive Bayes

$$y^* = \arg\max_c \log \pi_c + \sum_k z_k \log \theta_{ck}$$

For binary classification, we thus determine the label based on the sign of

$$\log \pi_1 + \sum_k z_k \log \theta_{1k} - \left(\log \pi_2 + \sum_k z_k \log \theta_{2k} \right)$$

This is just a linear function of the features $\{z_k\}$

$$w_0 + \sum_k z_k w_k$$

where we "absorb" $w_0 = \log \pi_1 - \log \pi_2$ and $w_k = \log \theta_{1k} - \log \theta_{2k}$.

Naive Bayes is a linear classifier

Fundamentally, what really matters in deciding decision boundary is

$$w_0 + \sum_k z_k w_k$$

This motivates many new methods. One of them is logistic regression, to be discussed in next lecture.