CS145: INTRODUCTION TO DATA MINING

5: Vector Data: Support Vector Machine

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Methods to Learn: Last Lecture

	Vector Data	Set Data	Sequence Data	Text Data
Classification	**Logistic Regression;** **Decision Tree**; KNN SVM; NN			Naïve Bayes for Text
Clustering	K-means; hierarchical clustering; DBSCAN; Mixture Models			PLSA
Prediction	**Linear Regression** GLM*			
Frequent Pattern Mining		Apriori; FP growth	GSP; PrefixSpan	
Similarity Search			DTW	

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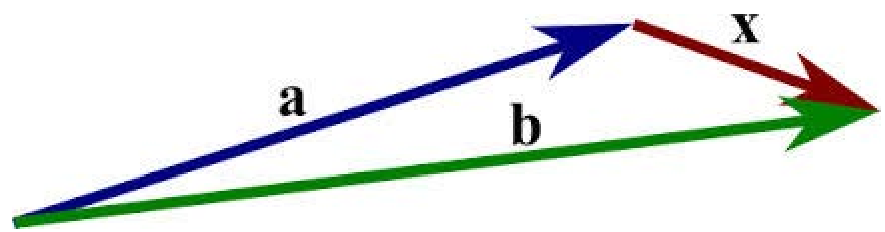
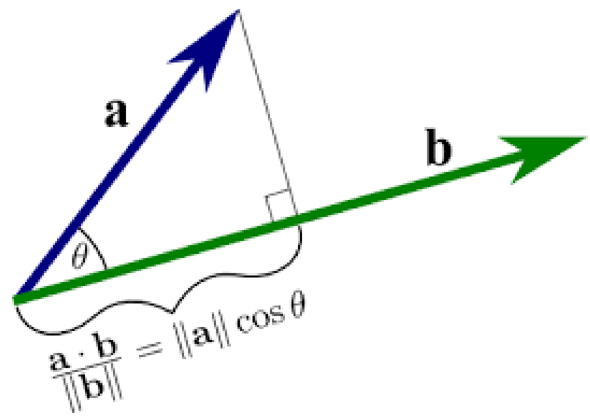
Support Vector Machine

- Introduction
- Linear SVM
- Non-linear SVM
- Scalability Issues*
- Summary

Math Review

- Vector
  - $\boldsymbol{x} = (\mathrm{x}_1, \mathrm{x}_2, \dots, x_n)$
  - Subtracting two vectors: $\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{a}$





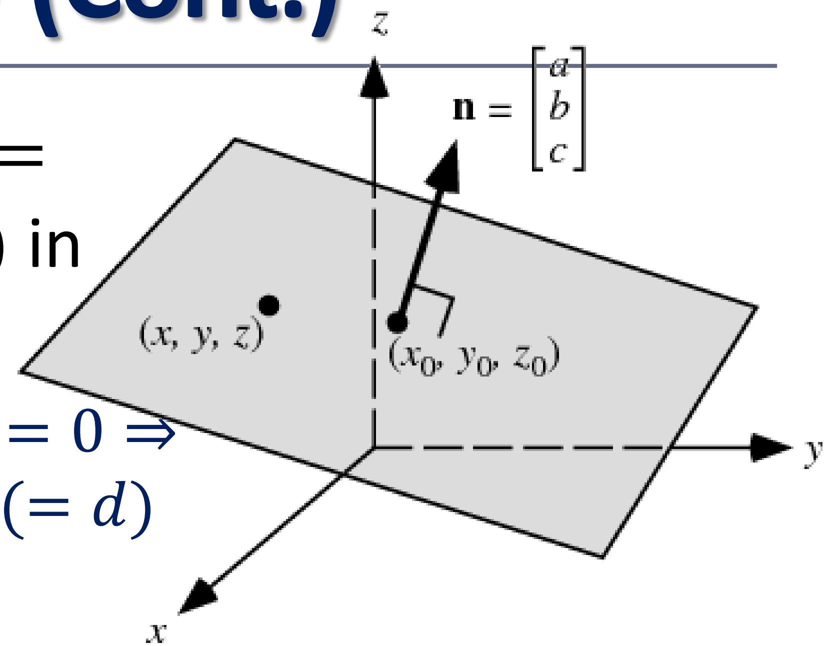
- Dot product
  - $\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$
  - Geometric interpretation: projection
  - If $\boldsymbol{a}$ *and* $\boldsymbol{b}$ are orthogonal, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Math Review (Cont.)

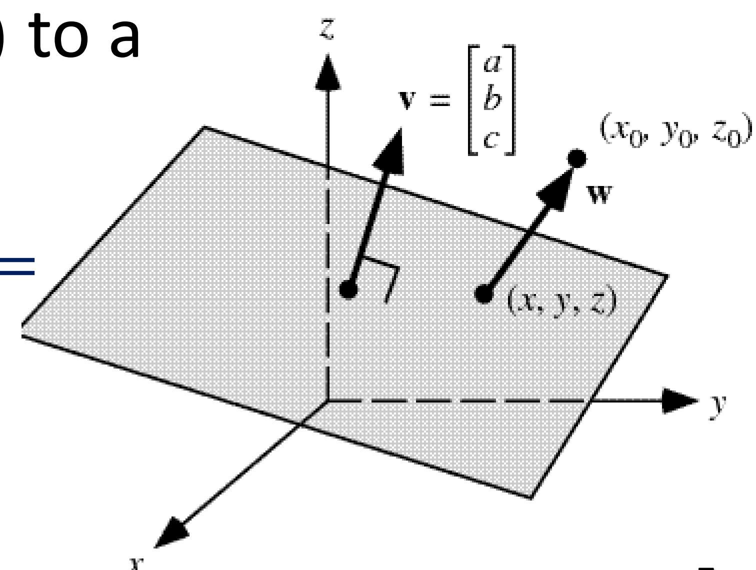
- Plane/Hyperplane
  - $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$
  - Line (n=2), plane (n=3), hyperplane (higher dimensions)
- Normal of a plane
  - $\boldsymbol{n} = (a_1, a_2, \ldots, a_n)$
  - a vector which is perpendicular to the surface

Math Review (Cont.)

- Define a plane using normal $\boldsymbol{n} = (a, b, c)$ and a point (x_0, y_0, z_0) in the plane:
  - $(a, b, c) \cdot (x_0 - x, y_0 - y, z_0 - z) = 0 \Rightarrow ax + by + cz = ax_0 + by_0 + cz_0 (= d)$



- Distance from a point (x_0, y_0, z_0) to a plane $ax + by + cz = \text{d}$
  - $\left|(x_0 - x, y_0 - y, z_0 - z) \cdot \frac{(a,b,c)}{||(a,b,c)||}\right| = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$



Linear Classifier

- Given a training dataset $\{\boldsymbol{x}_i, y_i\}_{i=1}^{N}$
  - A separating hyperplane can be written as a linear combination of attributes

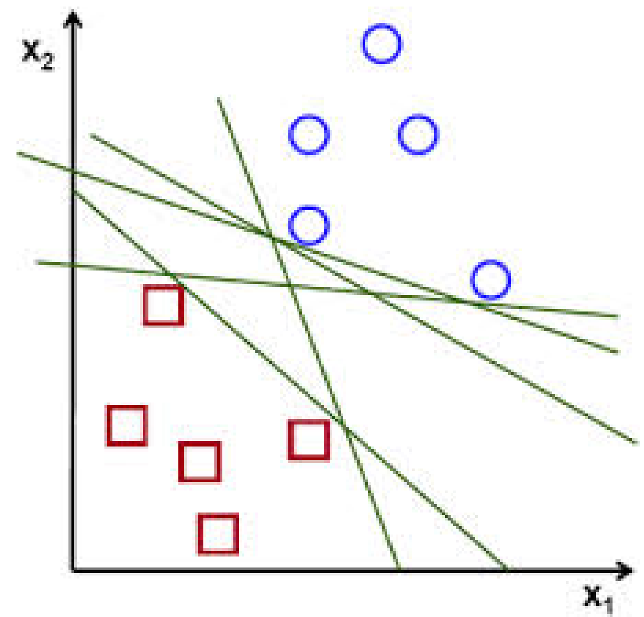
    $\mathbf{W} \bullet \mathbf{X} + b = 0$

    where $\mathbf{W}=\{w_1, w_2, \ldots, w_n\}$ is a weight vector and b a scalar (bias)
  - For 2-D it can be written as

    $w_0 + w_1 x_1 + w_2 x_2 = 0$
  - Classification:

    $w_0 + w_1 x_1 + w_2 x_2 > 0 \;\Rightarrow\; y_i = +1$

    $w_0 + w_1 x_1 + w_2 x_2 \leq 0 \;\Rightarrow\; y_i = -1$



Recall

- Is the decision boundary for logistic regression linear?

- Is the decision boundary for decision tree linear?

Simple Linear Classifier: Perceptron

$\mathbf{x} = (1, x_1, x_2, \ldots, x_d)^T$ $\quad$ $\mathbf{w} = (\omega_0, \omega_1, \omega_2, \ldots, \omega_d)^T$

$y = \{1, -1\}$ $\quad$ $\alpha \in (0, 1]$ (learning rate)

Initialize $\mathbf{w} = \mathbf{0}$ (can be any vector)

Repeat:

- For each training example $(\mathbf{x}_i, y_i)$:
  - Compute $\quad \hat{y}_i = \mathrm{sign}(\mathbf{w}^{\mathbf{T}}\mathbf{x_i})$
  - if $(y_i \neq \hat{y}_i)$ $\quad \mathbf{w} = \mathbf{w} + \alpha(y_i \mathbf{x_i})$

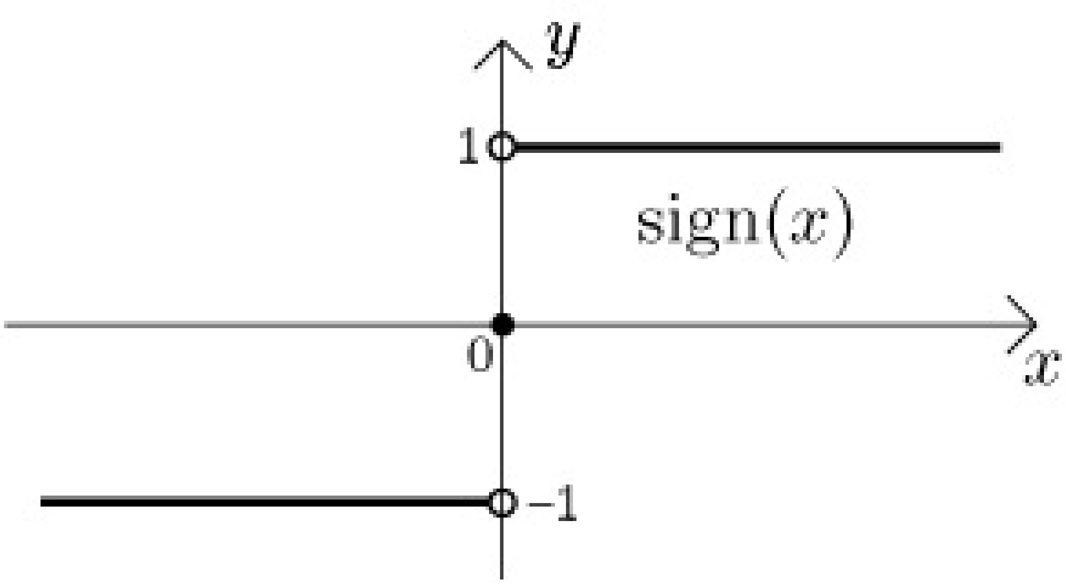
Until $(y_i = \hat{y}_i \quad \forall i = 1 \ldots N)$

Return $\mathbf{w}$

Loss function: $\max\{0, -y_i * w^T x_i\}$

More on Sign Function

- $$\operatorname{sign}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$



Example ($\alpha = 0.9$)

x0	x1	x2	true label	w before update	predicted label	w after update
1	0	1	Y	(0.0, 0.0, 0.0)	N	(0.9, 0.0, 0.9)
1	1	1	N	(0.9, 0.0, 0.9)	Y	(0.0, -0.9, 0.0)
1	0	0	Y	(0.0, -0.9, 0.0)	N	(0.9, -0.9, 0.0)
1	1	0	Y	(0.9, -0.9, 0.0)	N	(1.8, 0.0, 0.0)
1	0	1	Y	(1.8, 0.0, 0.0)	Y	(1.8, 0.0, 0.0)
1	1	1	N	(1.8, 0.0, 0.0)	Y	(0.9, -0.9, -0.9)
1	0	0	Y	(0.9, -0.9, -0.9)	Y	(0.9, -0.9, -0.9)
1	1	0	Y	(0.9, -0.9, -0.9)	N	(1.8, 0.0, -0.9)
1	0	1	Y	(1.8, 0.0, -0.9)	Y	(1.8, 0.0, -0.9)
1	1	1	N	(1.8, 0.0, -0.9)	Y	(0.9, -0.9, -1.8)
1	0	0	Y	(0.9, -0.9, -1.8)	Y	(0.9, -0.9, -1.8)
1	1	0	Y	(0.9, -0.9, -1.8)	N	(1.8, 0.0, -1.8)

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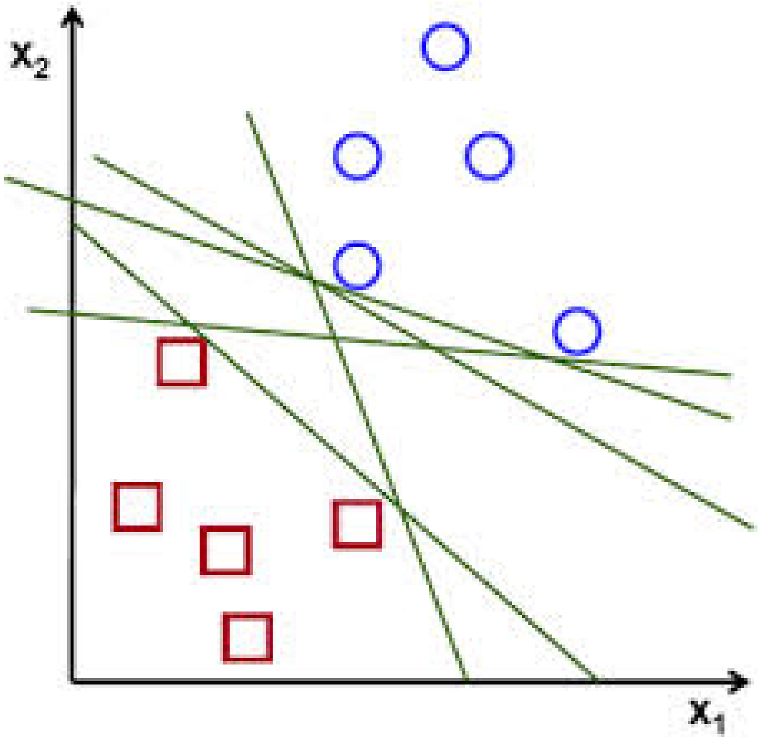
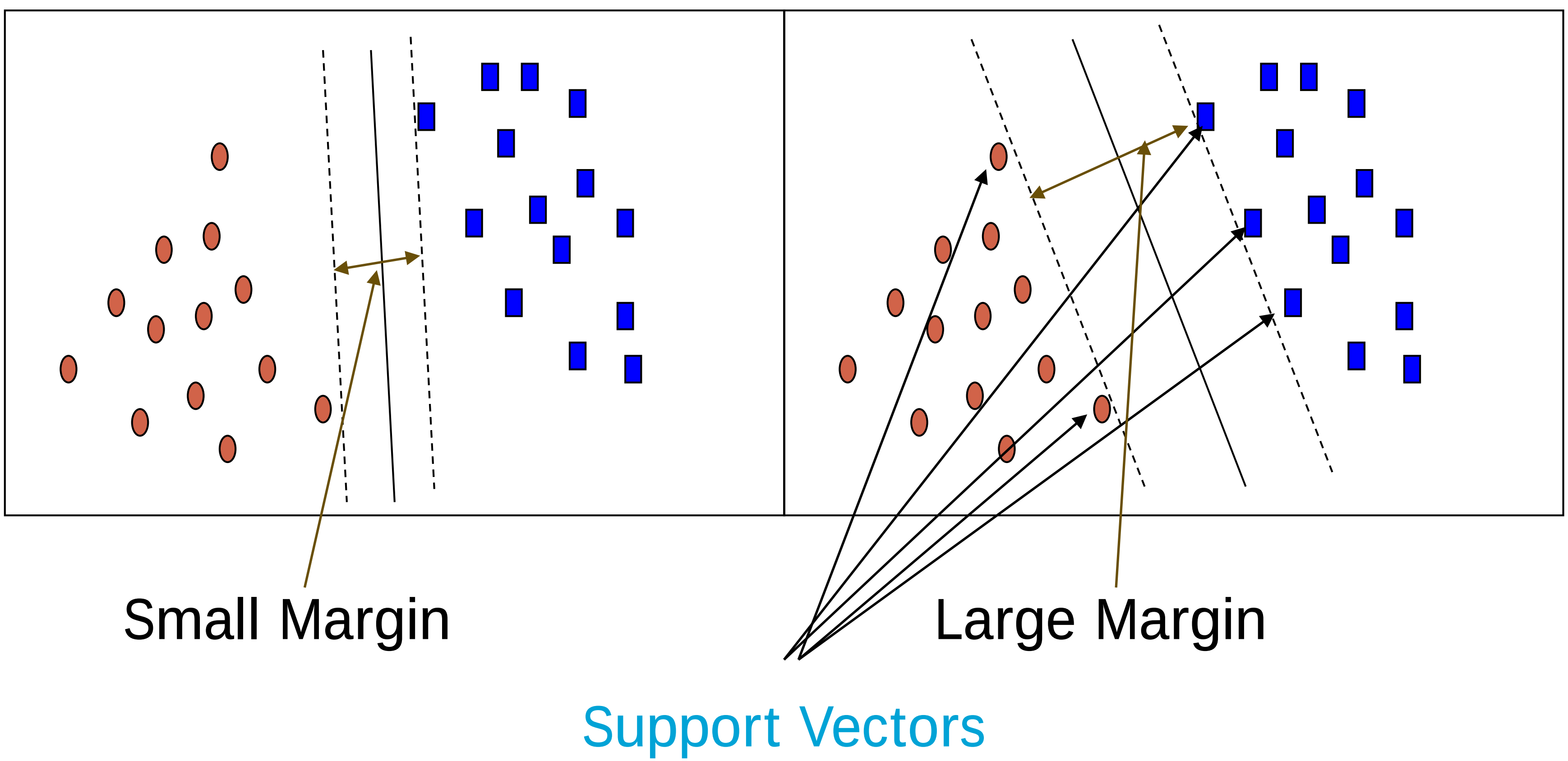
Can we do better?

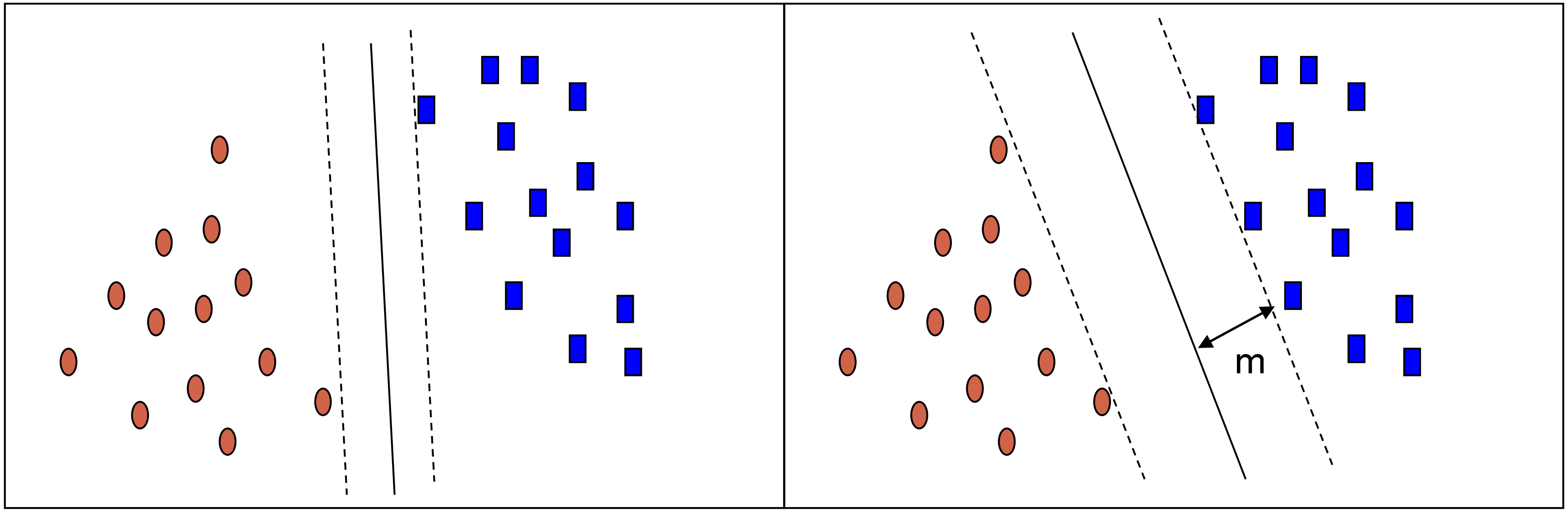
- Which hyperplane to choose?



SVM—Margins and Support Vectors



SVM—When Data Is Linearly Separable



Let data D be ($\mathbf{X}_1$, y_1), ..., ($\mathbf{X}_{|D|}$, $y_{|D|}$), where $\mathbf{X}_i$ is the set of training tuples associated with the class labels y_i

There are infinite lines (hyperplanes) separating the two classes but we want to find the best one (the one that minimizes classification error on unseen data)

SVM searches for the hyperplane with the largest margin, i.e., **maximum marginal hyperplane** (MMH)

SVM—Linearly Separable

- A separating hyperplane can be written as

  $$\mathbf{W} \bullet \mathbf{X} + b = 0$$

- The hyperplane defining the sides of the margin, e.g.,:

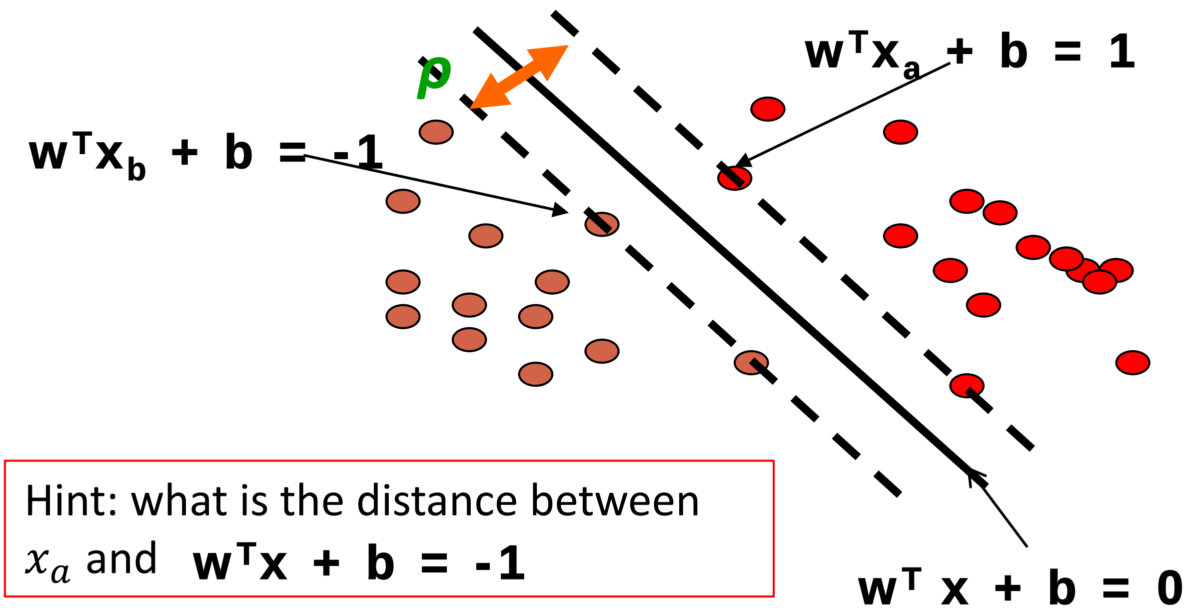
  H_1: $w_1 x_1 + w_2 x_2 + b \geq 1$ for $y_i = +1$, and

  H_2: $w_1 x_1 + w_2 x_2 + b \leq -1$ for $y_i = -1$

- Any training tuples that fall on hyperplanes H_1 or H_2 (i.e., the sides defining the margin) are **support vectors**
- This becomes a **constrained (convex) quadratic optimization** problem: Quadratic objective function and linear constraints → *Quadratic Programming (QP)* → Lagrangian multipliers

Maximum Margin Calculation

- **w**: decision hyperplane normal vector
- $\mathbf{x}_i$: data point *i*
- y_i: class of data point *i* (+1 or -1)



$$margin: \rho = \frac{2}{||\boldsymbol{w}||}$$

Hint: what is the distance between x_a and $\mathbf{w}^T\mathbf{x} + b = -1$

SVM as a Quadratic Programming

- QP

Objective: Find $\mathbf{w}$ and b such that $\rho = \frac{2}{||\boldsymbol{w}||}$ is maximized;

Constraints: For all $\{(\mathbf{x_i}, y_i)\}$

$\mathbf{w^T x_i} + b \geq 1$ if $y_i = 1$;

$\mathbf{w^T x_i} + b \leq -1$ if $y_i = -1$

- A better form

Objective: Find $\mathbf{w}$ and b such that $\mathbf{\Phi(w)} = ½\, \mathbf{w^T w}$ is minimized;

Constraints: for all $\{(\mathbf{x_i}, y_i)\}$: $y_i (\mathbf{w^T x_i} + b) \geq 1$

Solve QP

- This is now optimizing a *quadratic* function subject to *linear* constraints
- Quadratic optimization problems are a well-known class of mathematical programming problem, and many (intricate) algorithms exist for solving them (with many special ones built for SVMs)
- The solution involves constructing a *dual problem* where a *Lagrange multiplier* α_i is associated with every constraint in the primary problem:

Lagrange Formulation

- Introducing Lagrange multipliers $\alpha_i \geq 0$ for each constraint

Minimize

$$L(\mathbf{w}, b, \alpha) = \frac{1}{2}\mathbf{w}^{\mathsf{T}}\mathbf{w} - \sum_{i=1}^{N} \alpha_i (y_i(\mathbf{w}^{\mathsf{T}}\mathbf{x}_i + b) - 1)$$

Take the partial derivatives w.r.t $\mathbf{w}$, b:

$$\nabla_{\mathbf{w}} L = \mathbf{w} - \sum_{i=1}^{N} \alpha_i y_i \mathbf{x}_i = 0 \Longrightarrow \mathbf{w} = \sum_{i=1}^{N} \alpha_i y_i \mathbf{x}_i$$

$$\frac{\partial L}{\partial b} = -\sum_{i=1}^{N} \alpha_i y_i = 0$$

Primal Form and Dual Form

Primal

Objective: Find $\mathbf{w}$ and b such that $\boldsymbol{\Phi}(\mathbf{w}) = \frac{1}{2}\, \mathbf{w}^T\mathbf{w}$ is minimized;

Constraints: for all $\{(\mathbf{x_i}, y_i)\}$: $\quad y_i\, (\mathbf{w}^T\mathbf{x_i} + b) \geq 1$

Equivalent under some conditions; also $\boldsymbol{w}, b, \boldsymbol{\alpha}$ $\boldsymbol{satisify}$ KKT conditions

Dual

Objective: Find $\alpha_1 \ldots \alpha_n$ such that

$\mathbf{Q}(\boldsymbol{\alpha}) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j \mathbf{x}_i^T\mathbf{x}_j$ is maximized and

Constraints

(1) $\Sigma\alpha_i y_i = 0$

(2) $\alpha_i \geq 0$ for all α_i

- More derivations: http://cs229.stanford.edu/notes/cs229-notes3.pdf

The Optimization Problem Solution

- The solution has the form:

$$\mathbf{w} = \Sigma\alpha_i y_i \mathbf{x_i} \qquad b = y_k - \mathbf{w^T x_k} \text{ for any } \mathbf{x_k} \text{ such that } \alpha_k \neq 0$$

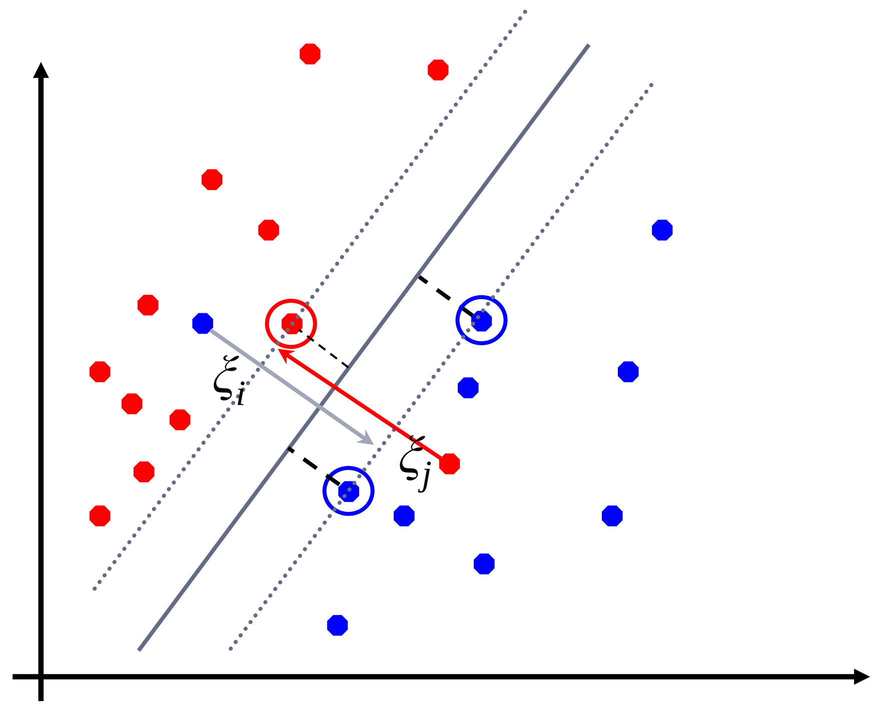
- Each non-zero α_i indicates that corresponding $\mathbf{x_i}$ is a support vector.
- Then the classifying function will have the form:

$$f(\mathbf{x}) = \Sigma\alpha_i y_i \mathbf{x_i}^{\mathbf{T}}\mathbf{x} + b$$

- Notice that it relies on an *inner product* between the test point $\mathbf{x}$ and the support vectors $\mathbf{x_i}$
  - We will return to this later.
- Also keep in mind that solving the optimization problem involved computing the inner products $\mathbf{x_i}^{\mathbf{T}}\mathbf{x_j}$ between all pairs of training points.

Soft Margin Classification

- If the training data is not linearly separable, *slack variables* ξ_i can be added to allow misclassification of difficult or noisy examples.
- Allow some errors
  - Let some points be moved to where they belong, at a cost
- Still, try to minimize training set errors, and to place hyperplane "far" from each class (large margin)



Soft Margin Classification Mathematically

- The old formulation:

> Find $\mathbf{w}$ and b such that
> $\boldsymbol{\Phi}(\mathbf{w}) = \frac{1}{2}\,\mathbf{w}^{\mathrm{T}}\mathbf{w}$ is minimized and for all $\{(\mathbf{x_i}, y_i)\}$
> $y_i\,(\mathbf{w^T x_i} + \mathrm{b}) \geq 1$

- The new formulation incorporating slack variables:

> Find $\mathbf{w}$ and b such that
> $\boldsymbol{\Phi}(\mathbf{w}) = \frac{1}{2}\,\mathbf{w}^{\mathrm{T}}\mathbf{w} + C\sum\xi_i$ is minimized and for all $\{(\mathbf{x_i}, y_i)\}$
> $y_i\,(\mathbf{w^T x_i} + b) \geq 1 - \xi_i$ and $\xi_i \geq 0$ for all i

- Parameter *C* can be viewed as a way to control overfitting
  - A regularization term (L1 regularization)

Soft Margin Classification – Solution

- The dual problem for soft margin classification:

> Find $\alpha_1 \ldots \alpha_N$ such that
> $\mathbf{Q}(\boldsymbol{\alpha}) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j \mathbf{x_i}^T\mathbf{x_j}$ is maximized and
> (1) $\Sigma\alpha_i y_i = 0$
> (2) $0 \leq \alpha_i \leq C$ for all α_i

- Neither slack variables ξ_i nor their Lagrange multipliers appear in the dual problem!
- Again, $\mathbf{x_i}$ with non-zero α_i will be **support vectors**.
  - If $0<\alpha_i<C$, $\xi_i=0$
  - If $\alpha_i=C$, $\xi_i>0$
- Solution to the problem is:

> $\mathbf{w} = \Sigma\alpha_i y_i \mathbf{x_i}$
> $b = y_k - \mathbf{w}^T\mathbf{x_k}$ for any $\mathbf{x_k}$ such that $0<\alpha_k<C$

w is not needed explicitly for classification!

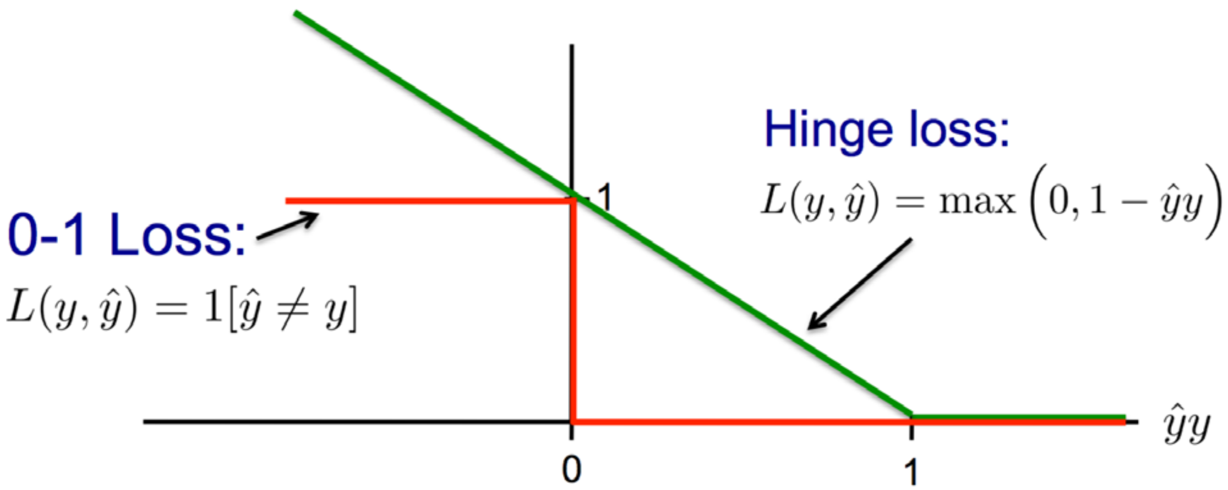
> $f(\mathbf{x}) = \Sigma\alpha_i y_i \mathbf{x_i}^T\mathbf{x} + b$

A Different View of Soft Margin SVM

- Hinge loss with regularization terms
  - $\mathbf{\Phi(w)} = ½\, \mathbf{w}^{\mathrm{T}}\mathbf{w} + C\Sigma\xi_i$

    $= ½\, \boxed{\mathbf{w}^{\mathrm{T}}\mathbf{w}} + C\Sigma \boxed{\max(0, 1- y_i\, (\mathbf{w}^{\mathrm{T}}\mathbf{x_i} + b))}$

L2 regularization **Hinge loss**



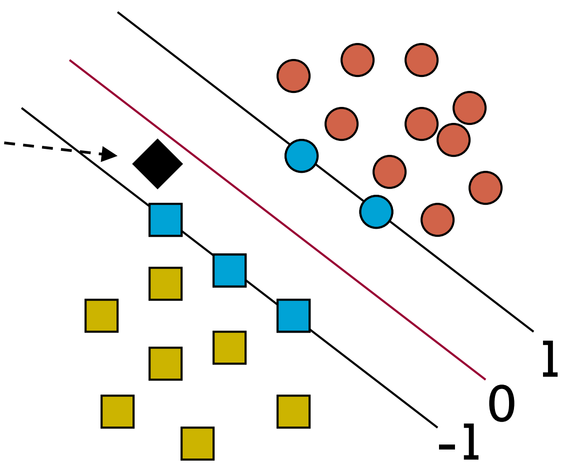
Classification with SVMs

- Given a new point **x**, we can score its projection onto the hyperplane normal:
  - I.e., compute score: $\mathbf{w}^T\mathbf{x} + b = \Sigma\alpha_i y_i \mathbf{x}_i^T\mathbf{x} + b$
    - Decide class based on whether < or > 0
  - Can set confidence threshold *t*.

Score > *t*: yes

Score < −*t*: no

Else: don't know



Linear SVMs: Summary

- The classifier is a *separating hyperplane.*
- The most "important" training points are the support vectors; they define the hyperplane.
- Quadratic optimization algorithms can identify which training points $\mathbf{x_i}$ are support vectors with non-zero Lagrangian multipliers α_i.
- Both in the dual formulation of the problem and in the solution, training points appear only inside inner products:

Find $\alpha_1 \ldots \alpha_N$ such that
$\mathbf{Q}(\boldsymbol{\alpha}) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j \boxed{\mathbf{x_i}^{\mathbf{T}}\mathbf{x_j}}$ is maximized and
(1) $\Sigma\alpha_i y_i = 0$
(2) $0 \leq \alpha_i \leq C$ for all α_i

$$f(\mathbf{x}) = \Sigma\alpha_i y_i \boxed{\mathbf{x_i}^{\mathbf{T}}\mathbf{x}} + b$$

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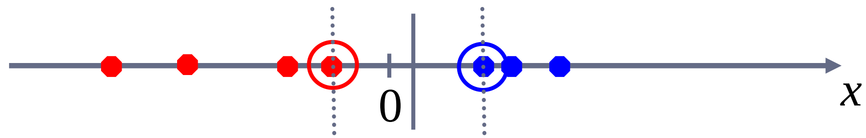
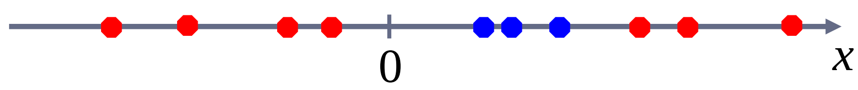
Non-linear SVMs

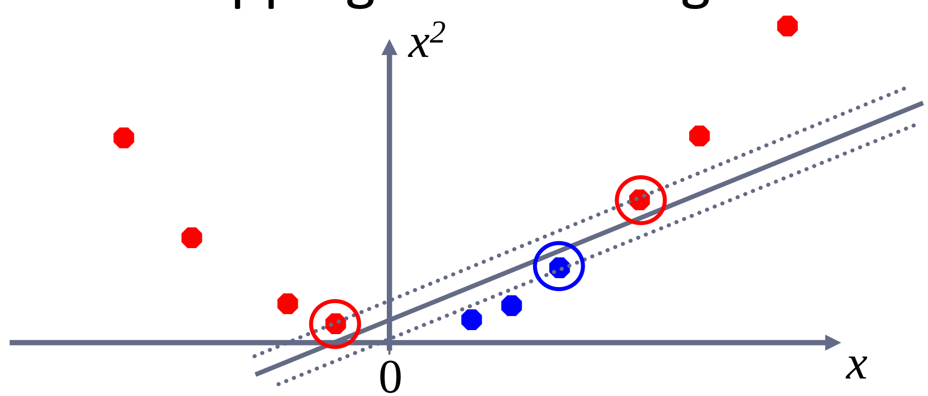
- Datasets that are linearly separable (with some noise) work out great:



- But what are we going to do if the dataset is just too hard?

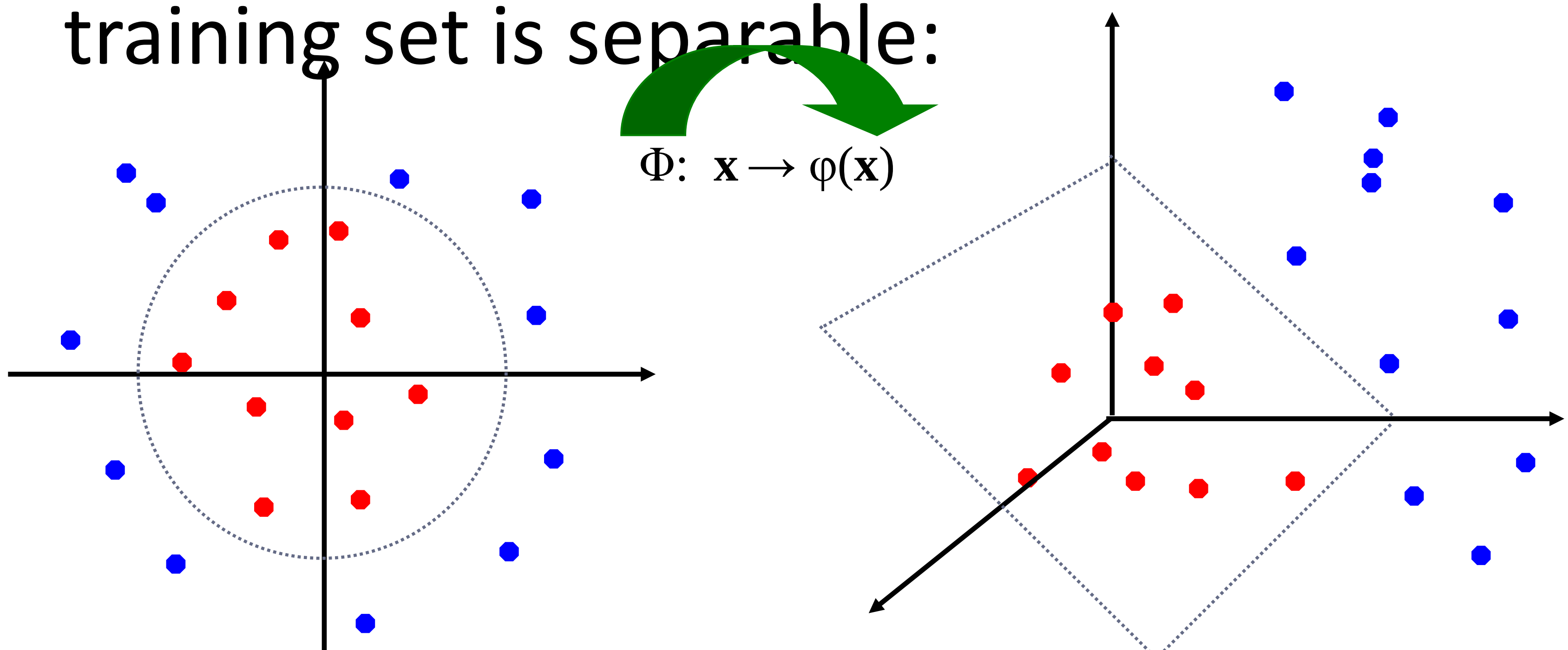


- How about … mapping data to a higher-dimensional space:



Non-linear SVMs: Feature spaces

- General idea: the original feature space can always be mapped to some higher-dimensional feature space where the training set is separable:



The "Kernel Trick"

- The linear classifier relies on an inner product between vectors $K(\mathbf{x}_i,\mathbf{x}_j)=\mathbf{x}_i^T\mathbf{x}_j$
- If every data point is mapped into high-dimensional space via some transformation Φ: $\mathbf{x} \rightarrow \phi(\mathbf{x})$, the inner product becomes:

$$K(\mathbf{x}_i,\mathbf{x}_j)= \phi(\mathbf{x}_i)^T\phi(\mathbf{x}_j)$$

- A *kernel function* is some function that corresponds to an inner product in some expanded feature space.

Example

- 2-dimensional vectors $\mathbf{x}=[x_1 \ \ x_2]$, let $K(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2$
- show that $K(\mathbf{x_i},\mathbf{x_j})= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$:

$K(\mathbf{x_i},\mathbf{x_j})=(1 + \mathbf{x_i}^T\mathbf{x_j})^2= 1+ x_{i1}^2x_{j1}^2 + 2\, x_{i1}x_{j1}\, x_{i2}x_{j2}+ x_{i2}^2x_{j2}^2 + 2x_{i1}x_{j1} + 2x_{i2}x_{j2}=$

$= [1 \ \ x_{i1}^2 \ \ \surd 2\, x_{i1}x_{i2} \ \ x_{i2}^2 \ \ \surd 2x_{i1} \ \ \surd 2x_{i2}]^T [1 \ \ x_{j1}^2 \ \ \surd 2\, x_{j1}x_{j2} \ \ x_{j2}^2 \ \ \surd 2x_{j1} \ \ \surd 2x_{j2}]$

$= \phi(\mathbf{x_i})^T\phi(\mathbf{x_j})$

where $\phi(\mathbf{x}) = [1 \ \ x_1^2 \ \ \surd 2\, x_1x_2 \ \ x_2^2 \ \ \surd 2x_1 \ \ \surd 2x_2]$

SVM: Different Kernel functions

- Instead of computing the dot product on the transformed data, it is math. equivalent to applying a kernel function $K(\mathbf{X}_i, \mathbf{X}_j)$ to the original data, i.e., $K(\mathbf{X}_i, \mathbf{X}_j) = \Phi(\mathbf{X}_i)^T\Phi(\mathbf{X}_j)$
- Typical Kernel Functions

$$\textbf{Polynomial kernel of degree } h: \quad K(X_i, X_j) = (X_i \cdot X_j + 1)^h$$

$$\textbf{Gaussian radial basis function kernel}: \quad K(X_i, X_j) = e^{-\|X_i - X_j\|^2/2\sigma^2}$$

$$\textbf{Sigmoid kernel}: \quad K(X_i, X_j) = \tanh(\kappa X_i \cdot X_j - \delta)$$

- *SVM can also be used for classifying multiple (> 2) classes and for regression analysis (with additional parameters)

Non-linear SVM

- Replace inner-product with kernel functions
  - Optimization problem

Find $\alpha_1 \ldots \alpha_N$ such that
$\mathbf{Q}(\boldsymbol{\alpha}) = \Sigma\alpha_i - \frac{1}{2}\Sigma\Sigma\alpha_i\alpha_j y_i y_j \mathbf{K}(\mathbf{x_i},\mathbf{x_j})$ is maximized and
(1) $\Sigma\alpha_i y_i = 0$
(2) $0 \leq \alpha_i \leq C$ for all α_i

  - Decision boundary

$f(\mathbf{x}) = \Sigma\alpha_i y_i \mathbf{K}(\mathbf{x_i},\mathbf{x}) + b$

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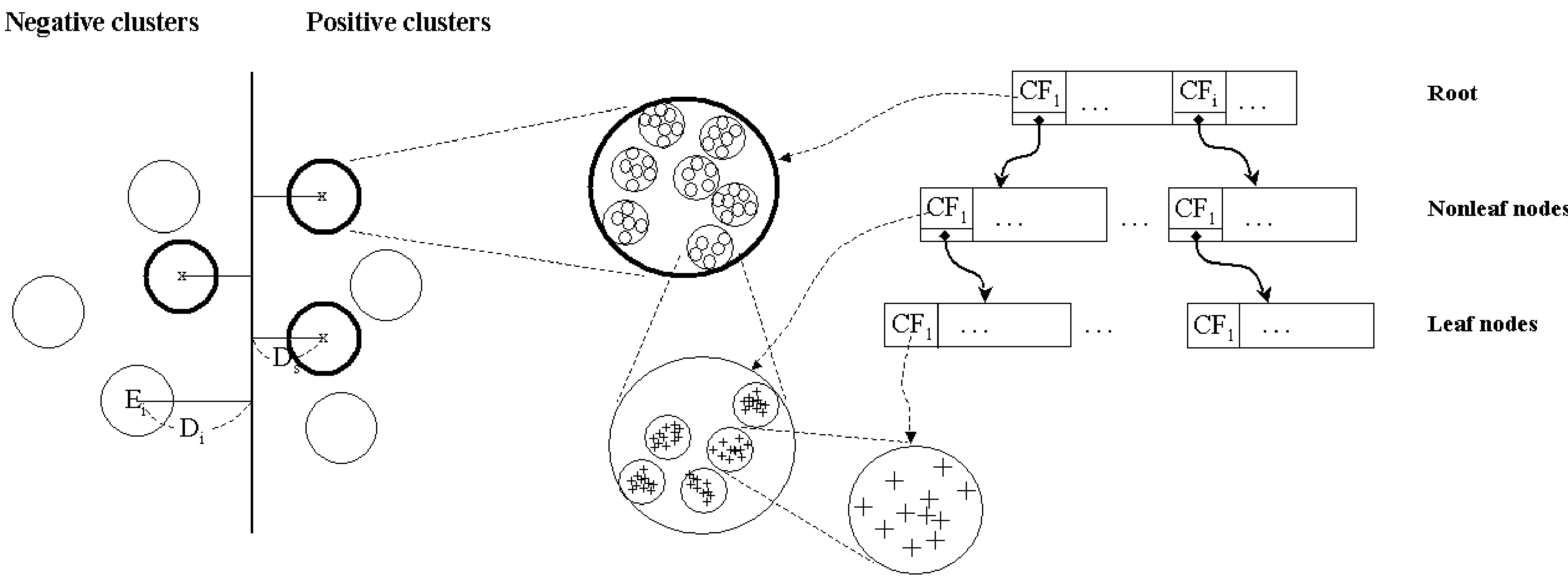
*Scaling SVM by Hierarchical Micro-Clustering

- SVM is not scalable to the number of data objects in terms of training time and memory usage
- H. Yu, J. Yang, and J. Han, "Classifying Large Data Sets Using SVM with Hierarchical Clusters", KDD'03)
- CB-SVM (Clustering-Based SVM)
  - Given limited amount of system resources (e.g., memory), maximize the SVM performance in terms of accuracy and the training speed
  - Use micro-clustering to effectively reduce the number of points to be considered
  - At deriving support vectors, de-cluster micro-clusters near "candidate vector" to ensure high classification accuracy

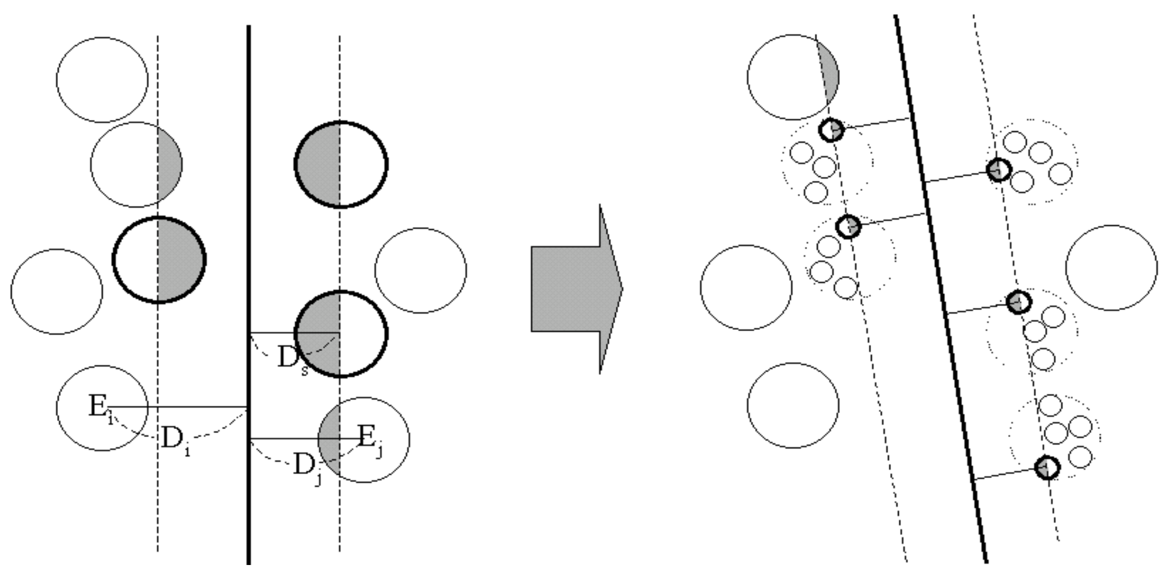
*CF-Tree: Hierarchical Micro-cluster



- Read the data set once, construct a statistical summary of the data (i.e., hierarchical clusters) given a limited amount of memory
- Micro-clustering: Hierarchical indexing structure
  - provide finer samples closer to the boundary and coarser samples farther from the boundary

*Selective Declustering: Ensure High Accuracy

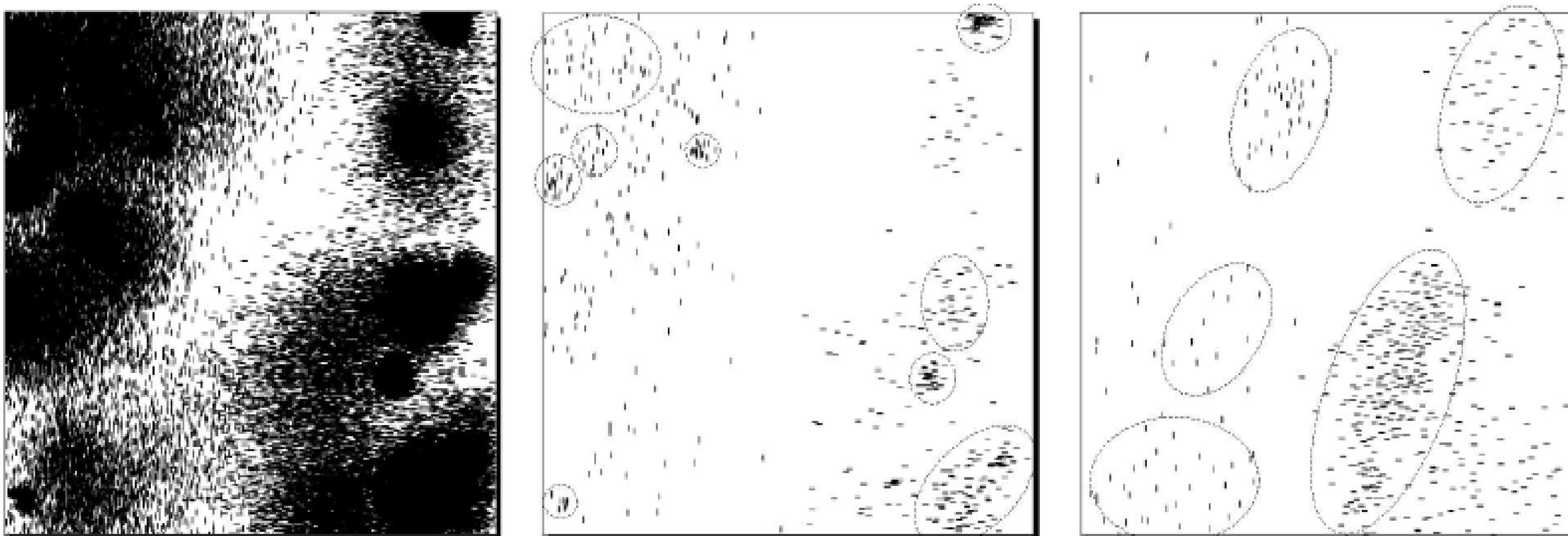
- CF tree is a suitable base structure for selective declustering
- De-cluster only the cluster E_i such that
  - $D_i - R_i < D_s$, where D_i is the distance from the boundary to the center point of E_i and R_i is the radius of E_i
  - Decluster only the cluster whose subclusters have possibilities to be the support cluster of the boundary
    - "Support cluster": The cluster whose centroid is a support vector



*CB-SVM Algorithm: Outline

- Construct two CF-trees from positive and negative data sets independently
  - Need one scan of the data set
- Train an SVM from the centroids of the root entries
- De-cluster the entries near the boundary into the next level
  - The children entries de-clustered from the parent entries are accumulated into the training set with the non-declustered parent entries
- Train an SVM again from the centroids of the entries in the training set
- Repeat until nothing is accumulated

*Accuracy and Scalability on Synthetic Dataset



(a) original data set ($N = 113601$)

(b) 0.5% randomly sampled data ($N = 603$)

(c) data distribution at the last iteration in CB-SVM ($N = 597$)

Figure 6: Synthetic data set in a two-dimensional space. '**|**': positive data; '**–**': negative data

- Experiments on large synthetic data sets shows better accuracy than random sampling approaches and far more scalable than the original SVM algorithm

Support Vector Machine

- Introduction
- Linear SVM
- Non-linear SVM
- Scalability Issues*
- Summary

Summary

- Support Vector Machine
  - Linear classifier; support vectors; kernel SVM

SVM Related Links

- SVM Website: http://www.kernel-machines.org/
- Representative implementations
  - **LIBSVM**: an efficient implementation of SVM, multi-class classifications, nu-SVM, one-class SVM, including also various interfaces with java, python, etc.
  - **SVM-light**: simpler but performance is not better than LIBSVM, support only binary classification and only in C
  - **SVM-torch**: another recent implementation also written in C
- From classification to regression and ranking:
  - http://www.dainf.ct.utfpr.edu.br/~kaestner/Mineracao/hwanjoyu-svmtutorial.pdf

More about Lagrangian

- Objective with equality constraints

$$\min_w f(w)$$
$$s.t.$$
$$h_i(w) = 0, for\ i = 1,2, \dots, l$$

- Lagrangian:
  - $L(w, \boldsymbol{\alpha}) = f(w) + \sum_i \alpha_i h_i(w)$
    - α_i: Lagrangian multipliers
- Solution: setting the derivatives of Lagrangian to be 0
  - $\frac{\partial L}{\partial w} = 0\ and\ \frac{\partial L}{\partial \alpha_i} = 0$ for every i

Generalized Lagrangian

- Objective with both equality and inequality constraints

$$\min_{w} f(w)$$
$$s.t.$$
$$h_i(w) = 0, for\ i = 1,2, \dots, l$$
$$g_j(w) \le 0, for\ j = 1,2, \dots, k$$

- Lagrangian
  - $L(w, \boldsymbol{\alpha}, \boldsymbol{\beta}) = f(w) + \sum_i \alpha_i h_i(w) + \sum_j \beta_j g_j(w)$
    - α_i: Lagrangian multipliers
    - $\beta_j \ge 0$: Lagrangian multipliers

Why It Works

- Consider function

$$\theta_p(w) = \max_{\alpha,\beta:\beta_j \geq 0} L(w, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- $\theta_p(w) = \begin{cases} f(w), & if\ w\ satisfies\ all\ constraints \\ \infty, & if\ w\ doesn't\ satisfy\ constraints \end{cases}$

- Therefore, minimize $f(w)$ with constraints is equivalent to minimize $\theta_p(w)$

Lagrange Duality

- The primal problem

$$p^* = \min_{w} \max_{\alpha,\beta:\beta_j \geq 0} L(w, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- The dual problem

$$d^* = \max_{\alpha,\beta:\beta_j \geq 0} \min_{w} L(w, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

- According to max-min inequality

$$p^* \leq d^*$$

  - When does equation hold?

Primal = Dual

- $p^* = d^*$, under some proper condition (Slater conditions)
  - f, g_j convex, h_i affine
  - Exists w, such that all $g_j(\mathrm{w}) < 0$
- (w^*, α^*, β^*) need to satisfy KKT conditions
  - $\frac{\partial L}{\partial w} = 0$
  - $\beta_j g_j(w) = 0$
  - $h_i(w) = 0, g_j(w) \leq 0, \beta_j \geq 0$