Polyhedral Compilation Foundations

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Feb 15, 2010

888.11, Class #4

Overview of Today's Lecture

Outline:

- Solution of the exercise
- Linear Programming (LP)
- Feautrier's scheduling algorithm
  - one-dimensional schedules
  - multidimensional schedules

Mathematical concepts:

- Linear Progamming
- (Parametric) Integer Programming

Checking the Legality of a Schedule

Exercise: given the dependence polyhedra, check if a schedule is legal

$$\mathcal{D}_1 : \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 \end{bmatrix} . \begin{pmatrix} eq \\ i_S \\ i'_S \\ n \\ 1 \end{pmatrix} \qquad \mathcal{D}_2 : \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 \end{bmatrix} . \begin{pmatrix} eq \\ i_S \\ i'_S \\ n \\ 1 \end{pmatrix}$$

1. $\Theta = i$
2. $\Theta = -i$

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1. $\Theta = i$
2. $\Theta = -i$

Solution: check for the emptiness of the polyhedron

$$\mathcal{P} : \begin{bmatrix} \mathcal{D} \\ i_S \succ i'_S \end{bmatrix} . \begin{pmatrix} i_S \\ i'_S \\ n \\ 1 \end{pmatrix}$$

where:

- $i_S \succ i'_S$ gets the consumer instances scheduled after the producer ones
- For $\Theta = -i$, it is $-i_S \succ -i'_S$, which is non-empty

Reminder from Last Week

Focused on one-dimensional schedules:

- A schedule is legal if the precedence condition is respected
- It is possible to build the set of legal 1-d schedules
  - Translate the problem as finding all non-negative functions over the dependence polyhedron
  - Model them thanks to the affine form of the Farkas Lemma
  - Proceed to identification + projection to get affine constraints on the schedule coefficients

Objective for a (good) scheduling strategy

- Output a legal schedule only
- Find one which maximizes/minimizes some criterion: objective function

Linear Programming (LP)

Definition (Linear Programming)

Linear Programming (LP) concerns the problem of maximizing or minimizing a real-valued function over a polyhedron.

$$\max\{cx \mid Ax \leq b\}$$

Theorem (Duality of Linear Programs)

Let A be a matrix, and let b and c be vectors. Then

$$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0,\ yA = c\}$$

Equivalent Formulations

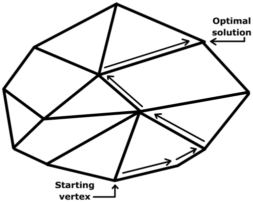
The following problems are equivalent:

1. $\max\{cx \mid Ax \le b\}$
2. $\max\{cx \mid x \ge 0,\ \ Ax \le b\}$
3. $\max\{cx \mid x \ge 0,\ \ Ax = b\}$
4. $\min\{cx \mid Ax \ge b\}$
5. $\min\{cx \mid x \ge 0,\ \ Ax \ge b\}$
6. $\min\{cx \mid x \ge 0,\ \ Ax = b\}$
7. $\min\{yb \mid yA \ge c\}$
8. ...

Solving a Linear Program: Simplex Algorithm

The most standard technique: Simplex [Dantzig]

- The hyperplane $cx = v$ contains the point where the objective function has value v
- Optimum v^* is the largest v such that $cx = v$ still intersects the polytope of feasible points
- The optimum is a face of the polytope
- Simplex: starts from a vertex, and build a "path" to reach the optimal vertex



Other techniques and Complexity results

- Worst-case complexity of Simplex: exponential time $O(2^n)$
- In practice, usually around $O(n^3)$
- Ellipsoid method [Khachiyan]: worst case $O(n^4)$
- Interior points methods [Karmarkar]: worst case $O(n^{3.5})$
- **LP admits a weakly polynomial-time algorithm**, so LP is in **P**

Applications to Polyhedral Optimization

- LP is for real-valued objective functions
- **But we mostly use integer coefficients**

- Refinement needed: Integer Linear Programming

- Even worse: we use parametric solution sets
- We often require **Parametric Integer Programming**

Integer Linear Programming (ILP)

- ILP requires the unkown variables to be integers
- Fundamental complexity change: **ILP is NP-hard**

- Several techniques to solve an ILP: branch-and-cut, branch-and-bound, cutting planes, ...

- **Most optimization problems in the polyhedral model are modeled as ILP**

  Examples: parallelization, locality, etc.

Parametric Integer Programming (PIP)

Parametric Integer Programming [Feautrier]:

- The feasible set is parametric
- The optimal solution may not be the same for different parameter values
- PIP: "parameterized" Simplex + Gomory cuts, finds the lexicographically smallest point of a parametric polyhedron
- **Output is a Quasi-Affine Solution Tree (QUAST)**

QUAST Example: if $M = 0$ then $\{x = 0\}$ else if $M \geq 1$ then $\{x = 42\}$

Using PIPLib

- Our standard tool to solve a PIP
- Uses the same convention as PolyLib: eq/ineq on first column
- PIPLib finds the lexicosmallest point in a parametric polyhedron
  - To encode a program, add extra variables at the beginning of the system
  - These will be minimized

A few facts to keep in mind:

- The order of variables in the PIP matters (lexico-smallest is found)
- The order of parameters matters (a different solution can be found)

Back to Scheduling: Feautrier's

Feautrier's 1-d scheduling algorithm:

- Objective: find maximal fine-grain parallelism
- In other words: express the program loop nest as (at most) one outer sequential loop enclosing parallel loops
- This problem is equivalent to minimizing the schedule latency

Exercise: Why are the two problems equivalent?

Objective Function

Idea: bound the latency of the schedule and minimize this bound

Theorem (Schedule latency bound)

If all domains are bounded, and if there exists at least one 1-d schedule Θ, then there exists at least one affine form in the structure parameters:

$$L = \vec{u}.\vec{n} + w$$

such that:

$$\forall \vec{x}_R,\ L - \Theta_R(\vec{x}_R) \geq 0$$

- Objective function: $\min\{\vec{u}, w \mid \vec{u}.\vec{n} + w - \Theta \geq 0\}$
- Subject to Θ is a legal schedule, and $\theta_i \geq 0$
- In many cases, it is equivalent to take the lexicosmallest point in the polytope of non-negative legal schedules

Example

$$\min\{\vec{u}, w \mid \vec{u}.\vec{n} + w - \Theta \geq 0\} : \Theta_R = 0,\ \Theta_S = k+1$$

Example

```
parfor (i = 0; i < N; ++i)
  parfor (j = 0; j < N; ++j)
    C[i][j] = 0;
for (k = 1; k < N + 1; ++k)
  parfor (i = 0; i < N; ++i)
    parfor (j = 0; j < N; ++j)
      C[i][j] += A[i][k-1] + B[k-1][j];
```

Multidimensional Scheduling

- **Some program does not have a legal 1-d schedule**
- It means, it's not possible to enforce the precedence condition for all dependences

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    s += s;
```

- Intuition: multidimensional time means nested time loops
- The precedence constraint needs to be adapted to multidimensional time

Dependence Satisfaction

Definition (Strong dependence satisfaction)

Given $\mathcal{D}_{R,S}$, the dependence is strongly satisfied at schedule level k if

$$\forall \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S}, \quad \Theta_k^S(\vec{x}_S) - \Theta_k^R(\vec{x}_R) \geq 1$$

Definition (Weak dependence satisfaction)

Given $\mathcal{D}_{R,S}$, the dependence is weakly satisfied at dimension k if

$$\begin{array}{ll} \forall \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S}, & \Theta_k^S(\vec{x}_S) - \Theta_k^R(\vec{x}_R) \geq 0 \\ \exists \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S}, & \Theta_k^S(\vec{x}_S) = \Theta_k^R(\vec{x}_R) \end{array}$$

Program Legality and Existence Results

- All dependence must be strongly satisfied for the program to be correct
- **Once a dependence is strongly satisfied at level k, it does not contribute to the constraints of level $k+i$**

- Unlike with 1-d schedules, it is always possible to build a legal multidimensional schedule for a SCoP [Feautrier]

Theorem (Existence of an affine schedule)

Every static control program has a multdidimensional affine schedule

Reformulation of the Precedence Condition

- We introduce variable $\delta_1^{\mathcal{D}_{R,S}}$ to model the dependence satisfaction
- Considering the first row of the scheduling matrices, to preserve the precedence relation we have:

$$\forall \mathcal{D}_{R,S},\ \forall \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S}, \quad \Theta_1^S(\vec{x}_S) - \Theta_1^R(\vec{x}_R) \geq \delta_1^{\mathcal{D}_{R,S}}$$
$$\delta_1^{\mathcal{D}_{R,S}} \in \{0, 1\}$$

Lemma (Semantics-preserving affine schedules)

Given a set of affine schedules $\Theta^R, \Theta^S \ldots$ of dimension m, the program semantics is preserved if:

$$\begin{aligned} & \forall \mathcal{D}_{R,S},\ \exists p \in \{1, \ldots, m\},\ \delta_p^{\mathcal{D}_{R,S}} = 1 \\ \wedge \quad & \forall j < p,\ \delta_j^{\mathcal{D}_{R,S}} = 0 \\ \wedge \quad & \forall j \leq p, \forall \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S},\ \Theta_p^S(\vec{x}_S) - \Theta_p^R(\vec{x}_R) \geq \delta_j^{\mathcal{D}_{R,S}} \end{aligned}$$

A Greedy Scheduling Algorithm

- Objective: maximize fine-grain parallelism
- Equivalent to strongly satisfying the maximum number of dependences at the current level
  - Find this set of schedules (objective 1)
  - Find the schedule with minimal latency in this set (objective 2)
- Proceeds greedily: removes all previously strongly solved dependence, and solve the problem for the next schedule dimension

Exercise: Write the objective function which maximizes the number of dependences strongly satisfied at a given schedule level k

A Greedy Scheduling Algorithm

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Exercise: Write the objective function which maximizes the number of dependences strongly satisfied at a given schedule level k

$$\max\{\sum_i \delta_k^i \mid \Theta_k^S(\vec{x}_S) - \Theta_k^R(\vec{x}_R) \geq \delta_k^{\mathcal{D}_{R,S}}\}$$

Some Interesting Properties

- Feautrier's greedy heuristic extracts the maximal amount of fine-grain parallelism [Vivien]

- The maximal set of dependences which can be strongly solved at a given schedule level is unique
- This is true only if you do not bound the schedule coefficients

- The set of constraints to select a schedule at a given level are independent
- This formulation does not allow to build an ILP which considers multiple schedule levels, requires instead to build greedy algorithm (e.g., PluTo)

Next Week

- Building the set of all legal multidimensional schedules
- Permutability, tiling and memory optimizations
- Likely the last lecture...

In 2 weeks, I would like to have a student present a paper:

- Bondhugula, CC'08
- Irigoin and Triolet, POPL'88
- Bastoul, PACT'04
- Trifunovic, PACT'09