Posets, Lattices, and Fixpoints

CS240B Notes

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Partial Orders

Let S be a set. A *relation R* on S is a subset of $S \times S$. We denote the fact that $(x,\ y) \in R$ by xRy.

Let $\preceq$ be a relation on a set S. Then $\preceq$ is a *partial order* if the following conditions are satisfied:

(a) $x \preceq x$,

(b) $x \preceq y$ and $y \preceq x$ imply $x = y$ and

(c) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$, for all $x,\ y,\ z \in S$.

We also use the notation $(S, \preceq)$ to denote the partial order $\preceq$ on S. S is often called a poset.

Example: Let S be a set and 2^S be the set of all subsets of S. Then $(2^S, \subseteq)$, with $\subseteq$ denoting set inclusion, is a partial order.

LUBs and GLBs for a partial order $(S, \preceq)$

- $a \in S$ is an *upper bound* of a subset X of S if $x \preceq a$, for all $x \in X$. Similarly, $b \in S$ is a *lower bound* of X if $b \preceq x$, for all $x \in X$.
- $a \in S$ is the *least upper bound* of a subset X of S if a is an upper bound of X and, for all upper bounds a' of X, we have $a \preceq a'$. Similarly, $b \in S$ is the *greatest lower bound* of a subset X of S if b is a lower bound of X and, for all lower bounds b' of X, we have $b' \preceq b$.
- The least upper bound of X is denoted by $lub(X)$; the greatest lower bound of X is denoted by $glb(X)$. $lub(X)$, when it exists, is unique—same for $glb(X)$.
- The glb or lub may not exist for every subset of a partially ordered L.

Lattices

- A partially ordered set L is called a *lattice* when $lub(\{a, b\})$ and $glb(\{a, b\})$ exist for every two elements, $a, b \in L$.
- If L is a lattice, then $glb(X)$ and $lub(X)$ exist for every *finite* subset $X \subseteq L$. However this conclusion does not hold when X is infinite.
- A lattice L, is a *complete* lattice, when it contains the $lub(X)$ and $glb(X)$ for every $X \subseteq L$. (Finite lattices are always complete—infinite lattices might not be complete.)
- $\top$ denotes the *top element*, $lub(L)$, and $\bot$ denotes the *bottom element*, $glb(L)$, of the complete lattice L.

Lattices–cont

- In the previous example, $(2^S, \subseteq)$ is a complete lattice. The top element is S and the bottom element is $\emptyset$.
- Let $(L, \preceq)$ be a lattice and $X \subseteq L$. We say X is *a total order* when for every pair $x, y \in L$ either $x \preceq y$ or $y \preceq x$
- Let $(L, \preceq)$ be a lattice. Every $(N, \preceq)$, such that $N \subseteq L$ is called a *sublattice* of $(L, \preceq)$.
- Let L be a lattice, and N be a totally ordered sublattice of L. If N contains its bottom, then it is called a *chain*. A chain of L is a totally ordered sublattice closed at the bottom.

Mappings

Let $(L, \preceq)$ be a complete lattice and $T : L \rightarrow L$ be a mapping.

- We say T is *monotonic* if $T(x) \preceq T(y)$, whenever $x \preceq y$.
- A mapping is called *continuous* if preserves lubs on chains: T is *continuous* if $T(lub(X)) = lub(T(X))$ for every chain X of L.

By taking $X = \{x, y\}$, we see that every continuous mapping is monotonic. However, the converse is not true.

Fixpoints

- $x = T(x)$ is called a *fixpoint equation*. The solutions of these equations are called fixpoints of T.
- Let $(L, \preceq)$ be a complete lattice and $T : L \longrightarrow L$ be a mapping: $y \in L$ is the *least fixpoint* of T if y is a fixpoint (that is, $T(y) = y$), and for all fixpoints z of T, we have $y \preceq z$. Similarly, we define the *greatest fixpoint*.
- **Theorem (Knaster/Tarski)**: Let $(L, \preceq)$ be a complete lattice and $T : L \longrightarrow L$ be monotonic. Then T has a least fixpoint, $lfp(T)$, and a greatest fixpoint, $gfp(T)$.

Fixpoints by Powers of T

A simple constructive characterization of least fixpoints exists for for mappings that are continuous:
The n-th power, of $T : L \rightarrow L$, denoted $T \uparrow n$ is defined as follows:

$$T \uparrow 0\ (x) \; = \; x$$

$$T \uparrow (n+1)\ (x) \; = \; T(T \uparrow n\ (x))$$

$$\ldots$$

$$T \uparrow \omega\ (x) \; = \; \mathbf{lub}(\{T \uparrow n\ (x) \mid n \geq 0\})$$

Fixpoints by Powers of T–cont

Theorem: Let L be a complete lattice and $T : L \rightarrow L$ be continuous. Then, $lfp(T) = T \uparrow \omega\,(\bot)$.

Thus, for continuous functions, the least fixpoint can be computed by starting from the bottom and iterating the application of T ad infinitum—or until the $n + 1$ power is identical to n-th one.