Fixpoint Semantics for Logic Programs

CS240B Notes

Notes based on Section 8.10 of *Advanced Database Systems*—Morgan Kaufmann, 1997

C. Zaniolo, March 2002

Immediate Consequence Operator

Rules can be viewed as mappings. Recursive rules define a *Fixpoint Equation*

For positive programs, the *Immediate Consequence Operator* T_P is defined as follows:

$$T_P(I) = \{A \in B_P \mid \exists r : A \leftarrow A_1, ..., A_n \in ground(P), \{A_1, ..., A_n\} \subseteq$$

Thus T_P is a mapping from Herbrand interpretations of P to Herbrand interpretations of P. For the ancestor program: $I = \{anc(anne, marc), parent(marc, silvia)\}$, and

$$T_P(I) = \{anc(marc, silvia), anc(anne, silvia), mother(anne, silvia), mother(anne, marc)\}$$

Least Fixpoint of T_P

We can view a program P as defining the following *fixpoint equation* over Herbrand interpretations:

$$I = T_P(I)$$

In general, a fixpoint equation might have no solution, one solution or several solutions.

Interpretations are subsets of B_P —i.e., elements of the power set $2^{|B_P|}$.

Now, $(2^{|B_P|}, \subseteq)$ is a complete lattice and T_P is monotonic.

Two Important Theorems

From the Knaster/Tarski's fixpoint theorem:

- Theorem: Let P be a definite clause program. There always exist a least fixpoint for T_P, denoted $lfp(T_P)$.

It is also easy to prove that:

- Theorem: Let P be a definite clause program. Then, $M_P = lfp(T_P)$.

Thus, given a positive program P, $M_P = lfp(T_P)$ defines its meaning.

Operational Semantics: Powers of T_P

$$T_P^{\uparrow 0}(I) = I$$

$$\cdots$$

$$T_P^{\uparrow n+1}(I) = T_P(T_P^{\uparrow n}(I))$$

Moreover, with ω denoting the first limit ordinal, we define:

$$T_P^{\uparrow \omega}(I) = \bigcup_{n \geq 0} \{T^{\uparrow n}(I) \mid n \geq 0\}$$

Of particular interest are the powers of T_P starting from the empty set, i.e., for $I = \emptyset$

Theorem:
If P is a positive program, $lfp(T_P) = T_P^{\uparrow \omega}(\emptyset)$.

Computation of $lfp(T_P) = T_P^{\uparrow\omega}(\emptyset)$

- The successive powers of T_P, form an ascending chain, since
  - $T_P^{\uparrow 0}(\emptyset) \subseteq T_P^{\uparrow 1}(\emptyset)$ (base), and
  - $T_P^{\uparrow n}(\emptyset) \subseteq T_P^{\uparrow n+1}(\emptyset)$ (induction)
- Moreover: $T_P^{\uparrow k+1}(\emptyset) = \bigcup_{n \leq k} T_P^{\uparrow n}(\emptyset)$, and if $T_P^{\uparrow n+1}(\emptyset) = T_P^{\uparrow n}(\emptyset)$, then $T_P^{\uparrow n}(\emptyset) = T_P^{\uparrow \omega}(\emptyset)$.
- Thus, the least fixpoint can be computed by starting from the bottom and iterating the application of T until no new atoms are obtained and the $(n+1)^{th}$ power is identical to the n^{th} one—if such a condition never occurs then we have an infinite computation.