Boosting

Professor Ameet Talwalkar

Slide Credit: Professor Fei Sha

Outline

Midterm

- Next Tuesday in class from 8am - 9:50am
- Completely closed-book (no notes allowed)
- Will include roughly 10 short answer questions and 2 long questions
  - Short questions should take no more than 5 minutes on average
  - Long questions should take no more than 20 minutes each
- Covers all material through (and including) this Thursday's lecture
  - Goal is to test conceptual understanding of the course material
  - Suggestion: carefully review lecture notes and problem sets
- Office hours / Section
  - me: Today 10:00am-11:00am (Boelter 4531F)
  - Amogh: Monday 11:30a - 12:30p; Friday 2:30p - 3:30p (Boelter 2432)
  - Nikos: Monday 3:30p - 4:30p; Wednesday 2:00p - 3:00p (Boelter 2432)
  - Section: Friday 12:00pm - 1:50pm (Kinsey Teaching Pavilion 1200B)

Other Announcements

- Nikos will be lecturing on Thursday about Neural Networks

Outline

Support vector machines (SVM)

Primal Formulation

$$\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \quad \frac{1}{2}\|\boldsymbol{w}\|_2^2 + C\sum_n \xi_n$$

$$\text{s.t.} \quad y_n[\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{x}_n) + b] \geq 1 - \xi_n \quad \text{and} \quad \xi_n \geq 0, \quad \forall\, n$$

Two equivalent interpretations

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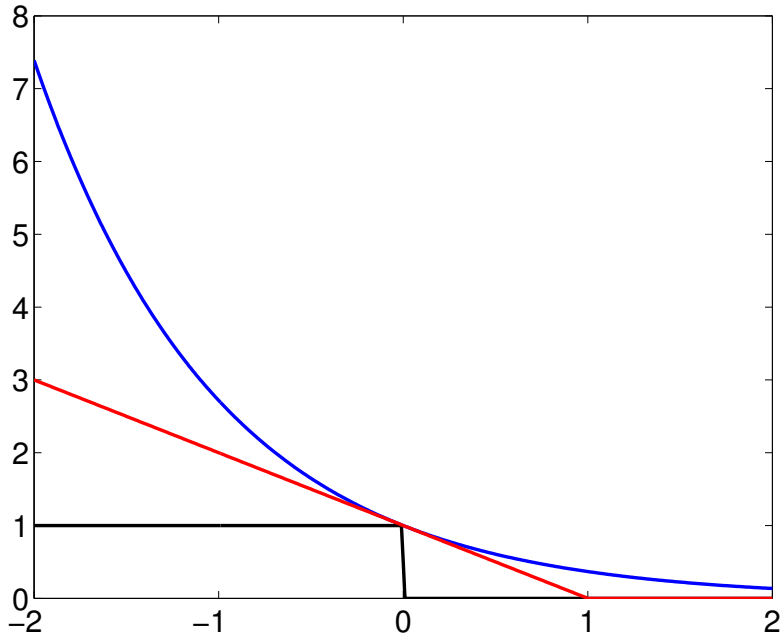
Two equivalent interpretations

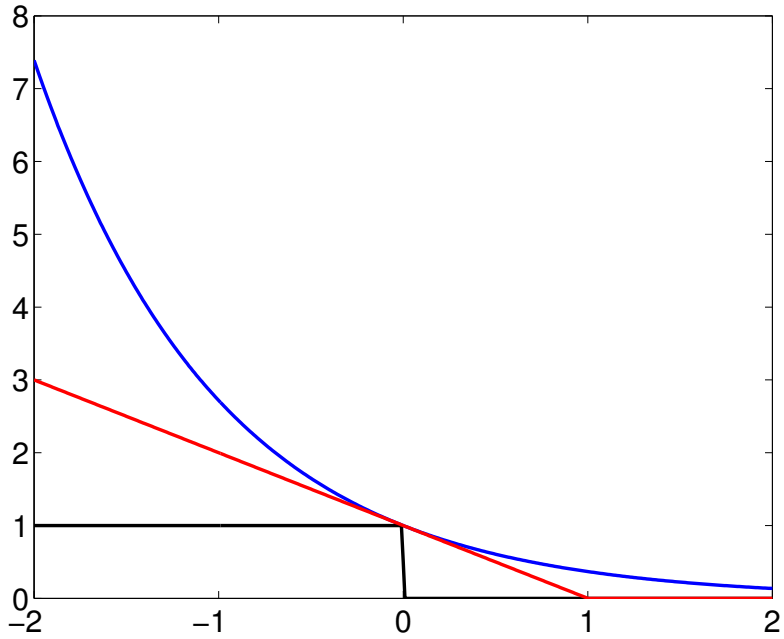
- Geometric: Maximizing (soft) margin
- Optimization: Minimize hinge loss with L2 regularization

Hinge Loss



- Upper-bound for $0/1$ loss function (black line)
- Convex surrogate to $0/1$ loss

Hinge Loss



- Upper-bound for $0/1$ loss function (black line)
- Convex surrogate to $0/1$ loss, though others exist as well
  - Hinge loss less sensitive to outliers than exponential (or logistic) loss
  - Logistic loss has a natural probabilistic interpretation
  - We can optimize exponential loss efficiently in a greedy manner (Adaboost)

Constrained Optimization

Primal Formulation

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- When working with constrained optimization problems with inequality constraints, we can write down primal and dual problems
- The dual solution is always a lower bound on the primal solution (weak duality)
- The duality gap equals 0 under certain conditions (strong duality), and in such cases we can either solve the primal or dual problem
- Strong duality holds for the SVM problem, and in particular the KKT conditions are necessary and sufficient for the optimal solution

Dual formulation of SVM and Kernel SVMs

$$\begin{aligned} \max_{\boldsymbol{\alpha}} \quad & \sum_n \alpha_n - \frac{1}{2} \sum_{m,n} y_m y_n \alpha_m \alpha_n \boldsymbol{\phi}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n) \\ \text{s.t.} \quad & 0 \leq \alpha_n \leq C, \quad \forall\ n \\ & \sum_n \alpha_n y_n = 0 \end{aligned}$$

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- Dual problem is also a convex quadratic programming involving N dual variables α_n
- Kernel SVM: replace inner products $\boldsymbol{\phi}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n)$ with a kernel function, $k(\boldsymbol{x}_m, \boldsymbol{x}_n)$

Recovering primal solution using dual variables

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- Using solely a kernel function, we can solve the dual optimization problem and make predictions at test time!
- Prediction only depends on support vectors, i.e., points with $\alpha_n > 0$!

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Weights

$$\boldsymbol{w} = \sum_n y_n \alpha_n \boldsymbol{\phi}(\boldsymbol{x}_n) \leftarrow \text{Linear combination of the input features}$$

Offset

$$b = [y_n - \boldsymbol{w}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}_n)] = [y_n - \sum_m y_m \alpha_m k(\boldsymbol{x}_m, \boldsymbol{x}_n)], \quad \text{for any } C > \alpha_n > 0$$

Prediction on a test point x

$$h(\boldsymbol{x}) = \text{SIGN}(\boldsymbol{w}^{\mathrm{T}} \boldsymbol{\phi}(\boldsymbol{x}) + b) = \text{SIGN}(\sum_n y_n \alpha_n k(\boldsymbol{x}_n, \boldsymbol{x}) + b)$$

Outline

Boosting

High-level idea: combine a lot of classifiers

- Sequentially construct / identify these classifiers one at a time
- Use *weak* classifiers to arrive at complex decision boundaries (*strong* classifiers)

Our plan

- Describe AdaBoost algorithm
- Derive the algorithm

Adaboost Algorithm

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- For $t = 1$ to T
  1. Train a weak classifier $h_t(\boldsymbol{x})$ using current weights $w_t(n)$, by minimizing the weighted classification error

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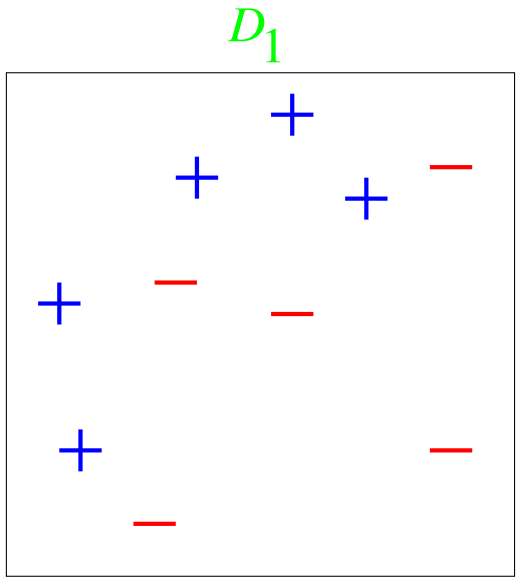
- Output the final classifier

$$h[\boldsymbol{x}] = \text{sign}\left[\sum_{t=1}^{T} \beta_t h_t(\boldsymbol{x})\right]$$

Example

10 data points and 2 features



- The data points are clearly not linear separable
- In the beginning, all data points have equal weights (the size of the data markers "+" or "-")

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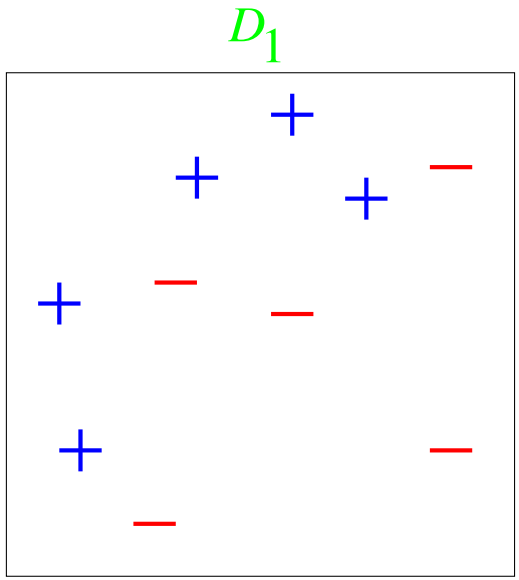

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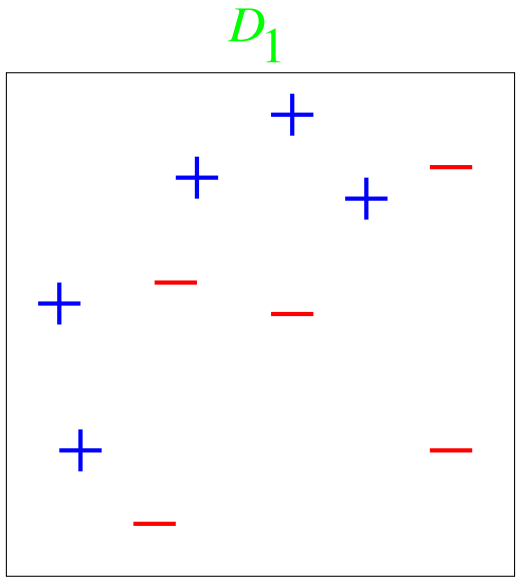

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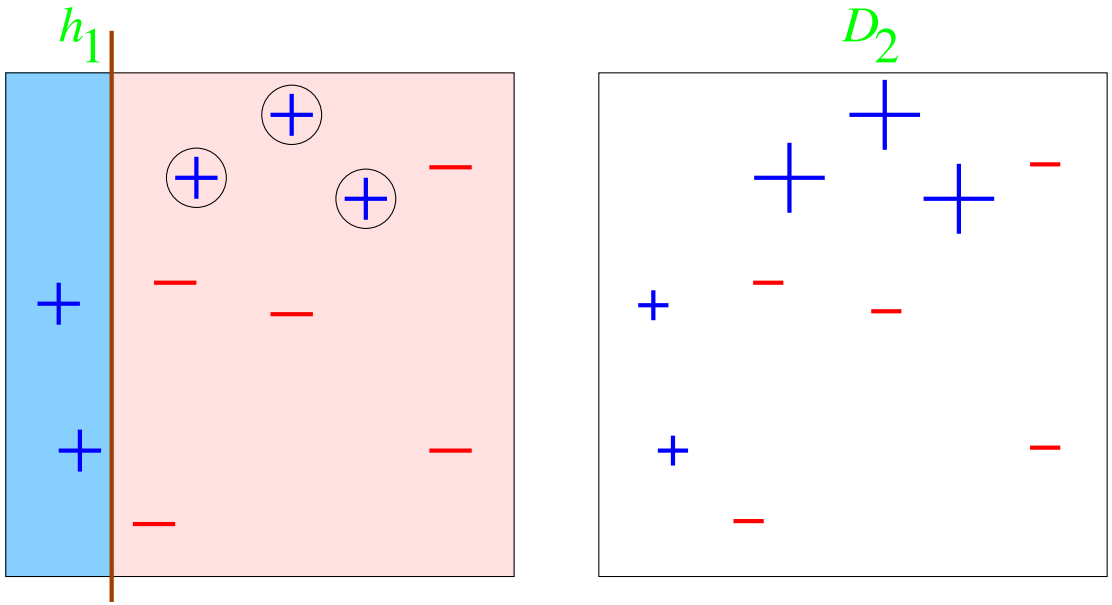
- In the beginning, all data points have equal weights (the size of the data markers "+" or "-")
- Base classifier $h(\cdot)$: either horizontal or vertical lines
  - These 'decision stumps' are just trees with a single internal node, i.e., they classifying data based on a single attribute

Round 1: $t = 1$



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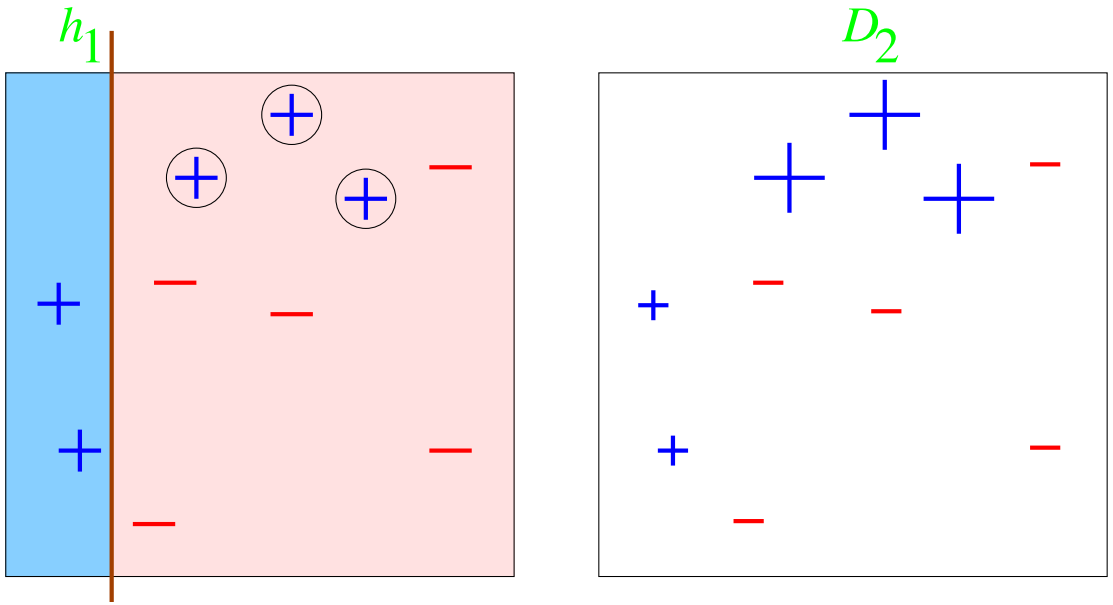
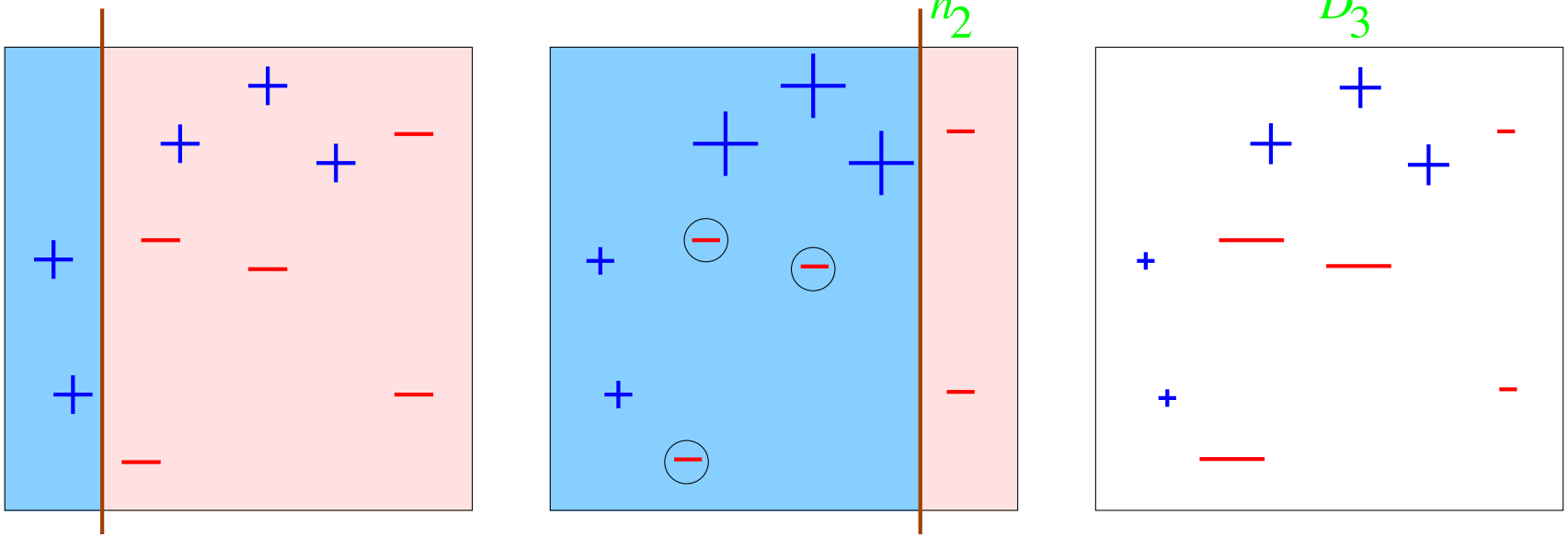


- 3 misclassified (with circles): $\epsilon_1 = 0.3 \rightarrow \beta_1 = 0.42$.
- Weights recomputed; the 3 misclassified data points receive larger weights

Round 2: $t = 2$



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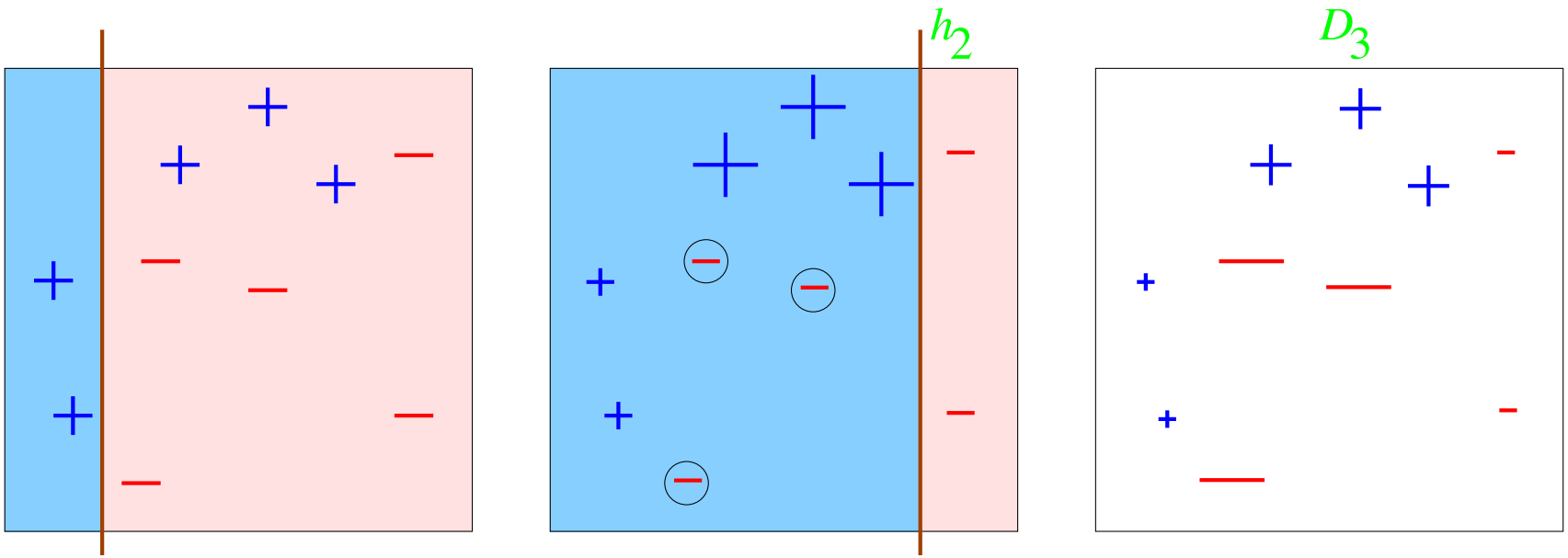
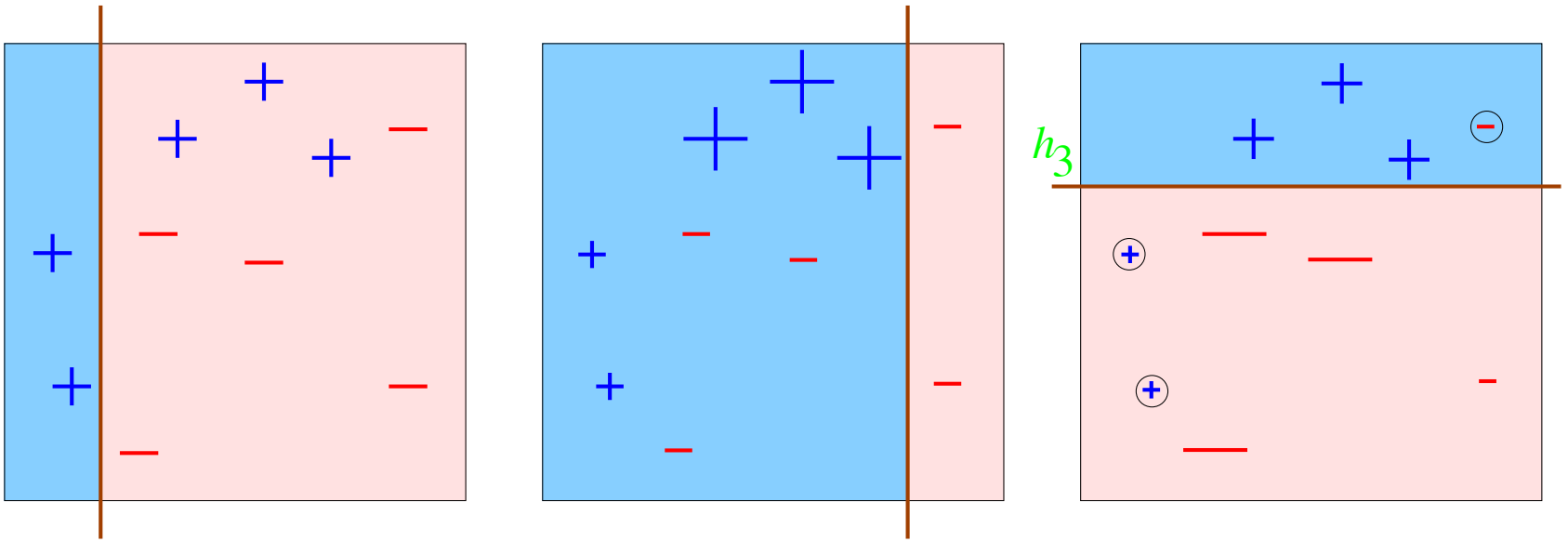


- 3 misclassified (with circles): $\epsilon_2 = 0.21 \rightarrow \beta_2 = 0.65$.
  Note that $\epsilon_2 \neq 0.3$ as those 3 data points have weights less than $1/10$
- 3 misclassified data points get larger weights
- Data points classified correctly in both rounds have small weights

Round 3: $t = 3$



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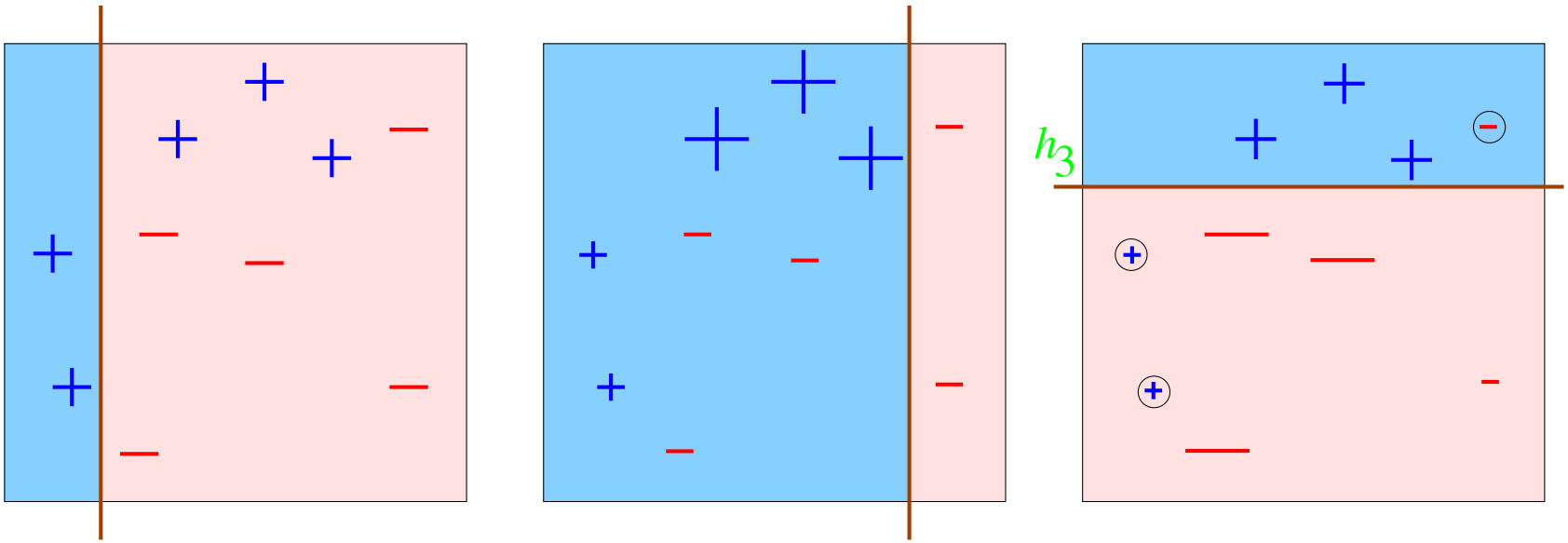

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- Previously correctly classified data points are now misclassified, hence our error is low; what's the intuition?

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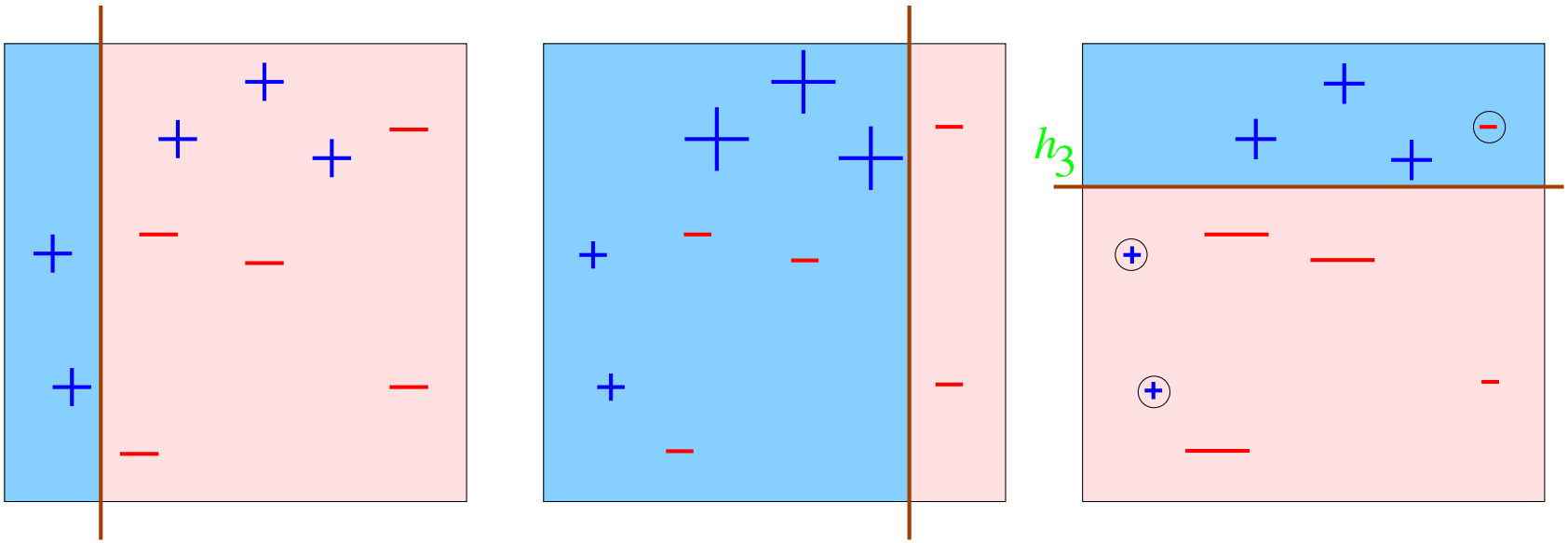
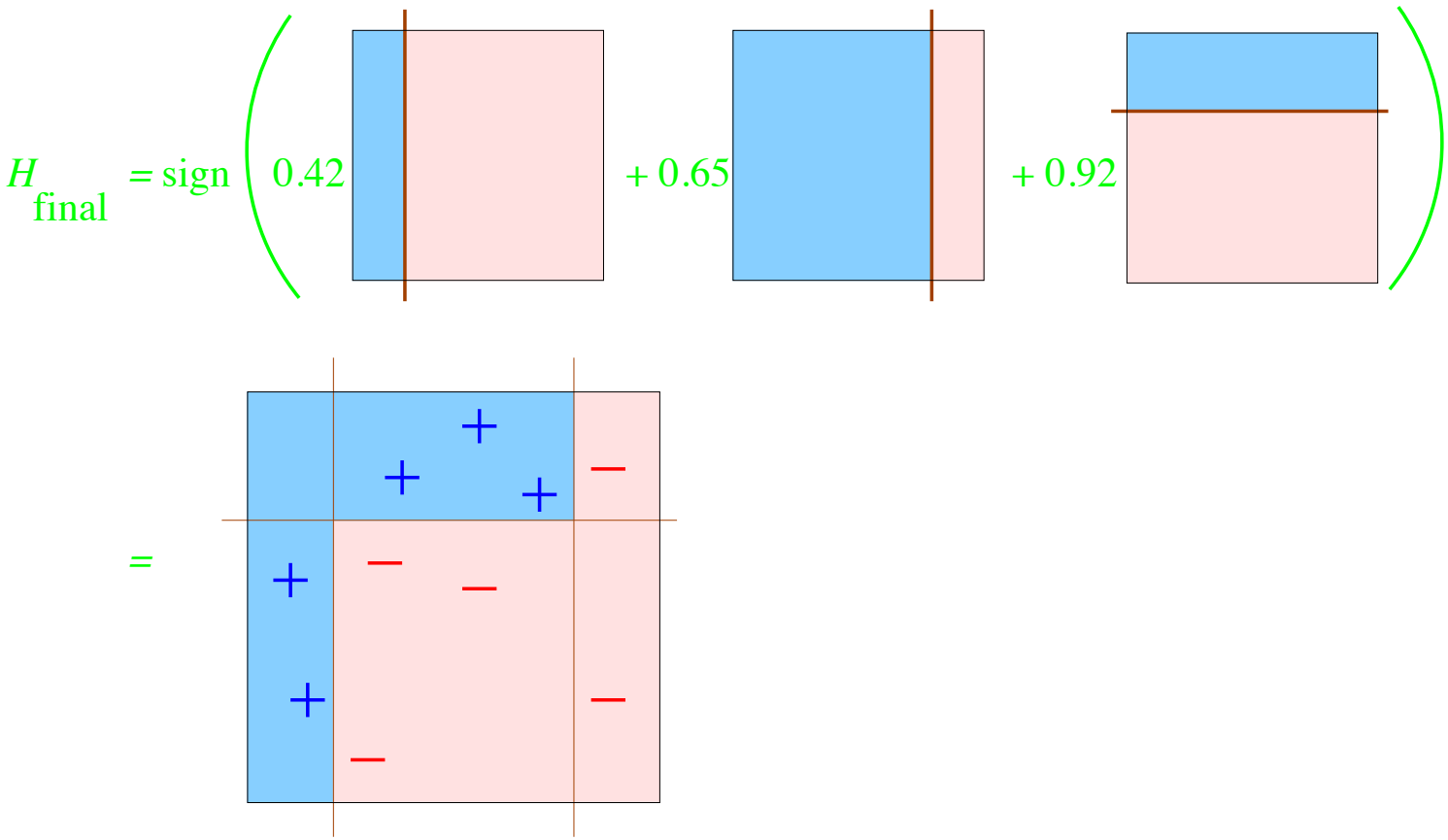


- 3 misclassified (with circles): $\epsilon_3 = 0.14 \rightarrow \beta_3 = 0.92$.
- Previously correctly classified data points are now misclassified, hence our error is low; what's the intuition?
  - Since they have been consistently classified correctly, this round's mistake will hopefully not have a huge impact on the overall prediction

Final classifier: combining 3 classifiers



- All data points are now classified correctly!

Why AdaBoost works?

It minimizes a loss function related to classification error.

Classification loss

- Suppose we want to have a classifier

$$h(\boldsymbol{x}) = \text{sign}[f(\boldsymbol{x})] = \begin{cases} 1 & \text{if } f(\boldsymbol{x}) > 0 \\ -1 & \text{if } f(\boldsymbol{x}) < 0 \end{cases}$$

- Our loss function is thus

$$\ell(h(\boldsymbol{x}), y) = \begin{cases} 0 & \text{if } yf(\boldsymbol{x}) > 0 \\ 1 & \text{if } yf(\boldsymbol{x}) < 0 \end{cases}$$

Namely, the function $f(\boldsymbol{x})$ and the target label y should have the same sign to avoid a loss of 1.

Surrogate loss

$0-1$ loss function $\ell(h(\boldsymbol{x}), y)$ is non-convex and difficult to optimize.
We can instead use a surrogate loss – what are examples?

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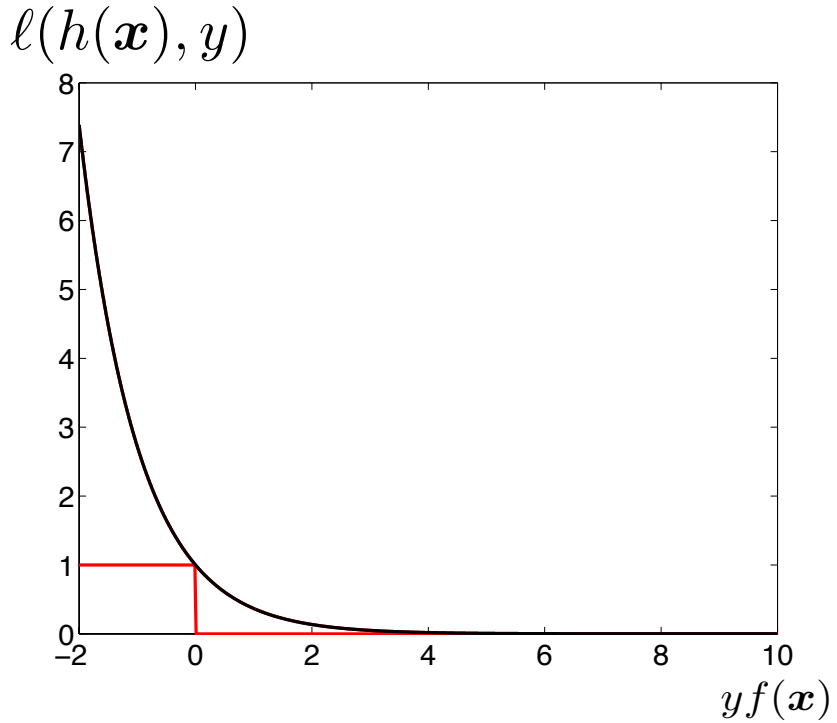
Exponential Loss

$$\ell^{\text{EXP}}(h(\boldsymbol{x}), y) = e^{-yf(\boldsymbol{x})}$$

$\ell^{\text{EXP}}(h(\boldsymbol{x}), y)$ is easier to handle numerically as it is differentiable



Choosing the t-th classifier

Suppose we have built a classifier $f_{t-1}(\boldsymbol{x})$, and we want to improve it by adding a weak learner $h_t(\boldsymbol{x})$

$$f(\boldsymbol{x}) = f_{t-1}(\boldsymbol{x}) + \beta_t h_t(\boldsymbol{x})$$

How can we choose optimally the new classifier $h_t(\boldsymbol{x})$ and the combination coefficient β_t?
Adaboost greedily *minimizes the exponential loss function*.

$$(h_t^*(\boldsymbol{x}), \beta_t^*) = \arg\min_{(h_t(\boldsymbol{x}), \beta_t)} \sum_n e^{-y_n f(\boldsymbol{x}_n)}$$

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where we have used $w_t(n)$ as a shorthand for $e^{-y_n f_{t-1}(\boldsymbol{x}_n)}$

The new classifier

We can decompose the *weighted* loss function into two parts

$$\sum_n w_t(n) e^{-y_n \beta_t h_t(\boldsymbol{x}_n)}$$

$$= \sum_n w_t(n) e^{\beta_t} \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + \sum_n w_t(n) e^{-\beta_t} \mathbb{I}[y_n = h_t(\boldsymbol{x}_n)]$$

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&= \sum_n w_t(n) e^{\beta_t} \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + \sum_n w_t(n) e^{-\beta_t} (1 - \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]) \\
&= (e^{\beta_t} - e^{-\beta_t}) \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + e^{-\beta_t} \sum_n w_t(n)
\end{aligned}$$

We have used the following properties to derive the above

- $y_n h_t(\boldsymbol{x}_n)$ is either 1 or -1 as $h_t(\boldsymbol{x}_n)$ is the output of a binary classifier
- The indicator function $\mathbb{I}[y_n = h_t(\boldsymbol{x}_n)]$ is either 0 or 1, so it equals $1 - \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$

Finding the optimal weak learner

Summary

$$(h_t^*(\boldsymbol{x}), \beta_t^*) = \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n w_t(n) e^{-y_n \beta_t h_t(\boldsymbol{x}_n)}$$

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What term(s) must we optimize to choose $h_t(\boldsymbol{x}_n)$?

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$$= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} (e^{\beta_t} - e^{-\beta_t}) \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$$

$$+ e^{-\beta_t} \sum_n w_t(n)$$

What term(s) must we optimize to choose $h_t(\boldsymbol{x}_n)$?

$$h_t^*(\boldsymbol{x}) = \arg\min_{h_t(\boldsymbol{x})} \epsilon_t = \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$$

Minimize weighted classification error as noted in step 1 of Adaboost!

How to choose β_t?

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We can do this by taking derivative with respect to β_t, setting to zero, and solving for β_t. After some calculation and using $\sum_n w_t(n) = 1$, we find:

$$\beta_t^* = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t}$$

which is precisely step 2 of Adaboost! (*Exercise – verify the solution*)

Updating the weights

Once we find the optimal weak learner we can update our classifier:

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$$= w_t(n) e^{-y_n \beta_t^* h_t^*(\boldsymbol{x}_n)} = \begin{cases} w_t(n) e^{\beta_t^*} & \text{if } y_n \neq h_t^*(\boldsymbol{x}_n) \\ w_t(n) e^{-\beta_t^*} & \text{if } y_n = h_t^*(\boldsymbol{x}_n) \end{cases}$$

Intuition Misclassified data points will get their weights increased, while correctly classified data points will get their weight decreased

Meta-Algorithm

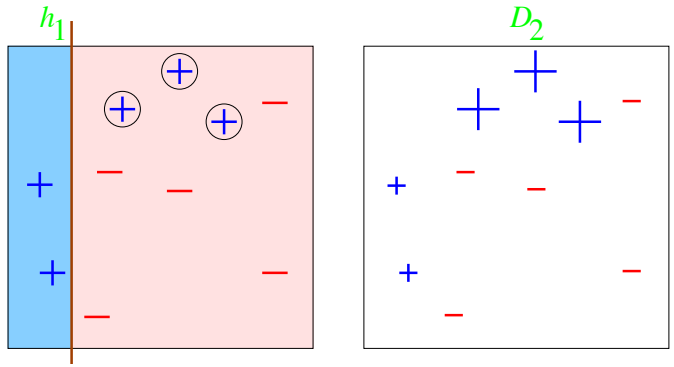
Note that the AdaBoost algorithm itself never specifies how we would get $h_t^*(\boldsymbol{x})$ as long as it minimizes the weighted classification error

$$\epsilon_t = \sum_n w_t(n)\mathbb{I}[y_n \neq h_t^*(\boldsymbol{x}_n)]$$

In this aspect, the AdaBoost algorithm is a meta-algorithm and can be used with any type of classifier

E.g., Decision Stumps

How do we choose the decision stump classifier given the weights at the second round of the following distribution?



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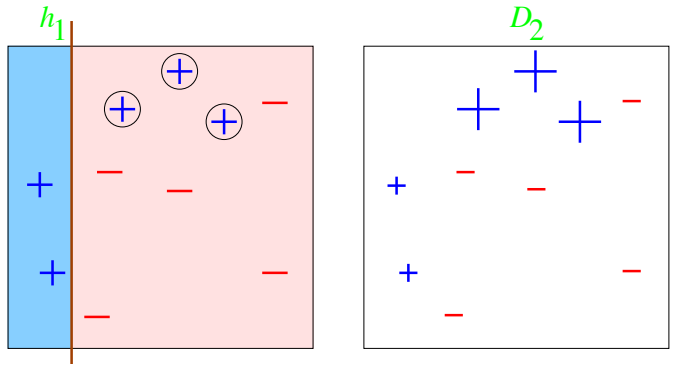

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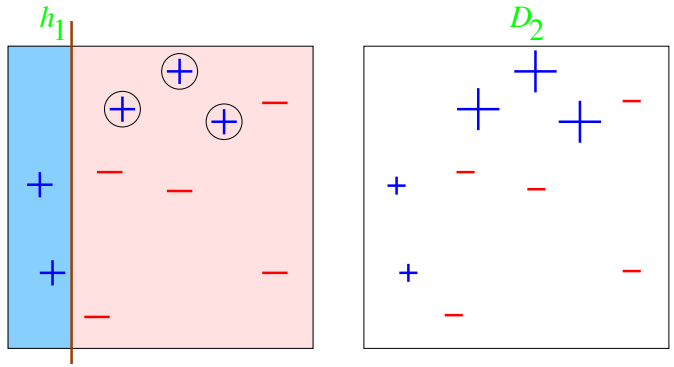

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- Presort data by each feature in $O(dN \log N)$ time
- Evaluate $N + 1$ thresholds for each feature at each round in $O(dN)$ time
- In total $O(dN \log N + dNT)$ time – this efficiency is an attractive quality of boosting!

Interpreting boosting as learning nonlinear basis

Two-stage process

- Get $\text{SIGN}[h_1(\boldsymbol{x})]$, $\text{SIGN}[h_2(\boldsymbol{x})]$,$\cdots$,
- Combine into a linear classification model

$$y = \text{SIGN}\left\{\sum_t \beta_t \text{SIGN}[h_t(\boldsymbol{x})]\right\} = \boldsymbol{\beta}^\top \boldsymbol{\phi}(\boldsymbol{x})$$

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In other words, each stage learns a nonlinear basis $\phi_t(\boldsymbol{x}) = \text{SIGN}[h_t(\boldsymbol{x})]$

- This is an alternative way to introduce non-linearity aside from kernel methods
- We could also try to learn the basis functions and the classifier at the same time, as we'll talk about with neural networks next class