Probabilistic Program Abstractions

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Abstract

Abstraction is a fundamental tool for reasoning about complex systems. Program abstraction has been utilized to great effect for analyzing deterministic programs. At the heart of program abstraction is the relationship between a concrete program, which is difficult to analyze, and an abstraction, which is more tractable. We generalize non-deterministic program abstractions to probabilistic program abstractions by explicitly quantifying the non-deterministic choices made by traditional program abstractions. We upgrade key theoretical program abstraction insights to the probabilistic context. Probabilistic program abstractions provide avenues for utilizing abstraction techniques from the programming languages community to improve the analysis of probabilistic programs.

1 INTRODUCTION & MOTIVATION

Program abstractions are a richly studied method from the programming languages community for reasoning about intractably complex programs [Cousot and Cousot, 1977]. An abstraction is typically an over-approximation to a program: any execution which is possible in the original program is contained within the abstraction. Over-approximation allows abstractions to be used to prove program invariants: any property of all executions in the abstraction is also true of all executions in the original program. To achieve this property while being more tractable than the concrete program, abstractions work on a simplified domain. The abstraction selectively models particular properties of the original program while utilizing non-determinism to conservatively model the rest.

A non-deterministic abstraction is useful for verifying reachability properties of a concrete program, but fails to support more nuanced queries such as probabilistic reachability, or more generally towards probabilistic program inference. We seek to enhance this representation by explicitly quantifying the non-deterministic choices made in the abstraction with probabilistic choices, making the program abstraction itself a probabilistic model.

The key contribution of this paper is the development of a foundational theory for extending non-deterministic program abstractions to probabilistic program abstractions. We show that probabilistic program abstractions, a generalized program abstraction with probabilistic instead of non-deterministic semantics, are defined by probabilistic programs with properties analogous to non-deterministic over-approximation.

A well-known construction of non-deterministic program abstractions is that of predicate abstractions [Graf and Saïdi, 1997; Ball et al., 2001]. Predicate abstractions are a popular style of program abstraction in the programming languages community that induces an abstraction relative to a given set of Boolean predicates about the program state. We construct a particular example of a probabilistic program abstraction, probabilistic predicate abstractions, via a novel construction of predicate abstractions which makes the non-deterministic choices syntactically explicit. This construction provides a straightforward avenue for generalizing predicate abstractions to probabilistic predicate abstractions by the introduction of a Bernoulli flip operator, showing that probabilistic predicate abstractions can be represented as a simple Bernoulli probabilistic programming language. Ultimately the construction of a probabilistic predicate abstraction, and probabilistic program abstractions in general, provide a mechanism for constructing a statistical model which is capable of representing distributions over the concrete program, similar to how non-deterministic program abstractions capture properties of all executions of a concrete program.
A concrete program is a syntactic object written \( \mathcal{C} \). The semantics of a concrete program, which for simplicity we also denote \( \mathcal{C} \), is a function from input states to output states over some concrete domain \( \mathcal{D}_C \). Concrete states are total assignments to all variables in the concrete domain, which we denote \( z \in \mathcal{D}_C \).

In general, the problem of proving that a given program satisfies a desired invariant is undecidable. Advances in theorem proving techniques such as Satisfiability Modulo Theories (SMT) solvers (De Moura and Bjørner, 2008) render reasoning in many useful theories tractable, yet there exist common program structures that lie outside of supported theories.

The framework of abstract interpretation (Cousot and Cousot, 1977), provides a general technique for relating a concrete program \( \mathcal{C} \) to another program \( \mathcal{A} \) which we refer to as an abstraction. We describe a specialization of the abstract interpretation framework.

**Definition 2.1. Abstract semantics of an abstraction.**

The abstract semantics of an abstraction \( \mathcal{A} \), which for simplicity we also denote \( \mathcal{A} \), is a function from input states to sets of output states over an abstract domain \( \mathcal{D}_A \), written \( \mathcal{A} : \mathcal{D}_A \to 2^{\mathcal{D}_A} \).

Intuitively, the nondeterminism in the abstract semantics of an abstraction represents uncertainty due to the loss of information in abstracting \( \mathcal{C} \) to \( \mathcal{A} \). We represent this non-determinism as a set of possible abstract states, denoted \( a \in \mathcal{D}_A \). To relate concrete programs with abstractions we introduce two mappings between concrete and abstract states.

**Definition 2.2. Abstraction and concretization functions.**

An abstraction function for \( \mathcal{D}_C \) and \( \mathcal{D}_A \) is a function \( \alpha : \mathcal{D}_C \to \mathcal{D}_A \) that maps each concrete state to its abstract representative. A concretization function for \( \mathcal{D}_C \) and \( \mathcal{D}_A \) is a function \( \gamma : \mathcal{D}_A \to 2^{\mathcal{D}_C} \) that maps each abstract state to a set of concrete states. When applied to sets, \( \gamma \) and \( \alpha \) respectively concretize or abstract each element of the set.

Abstraction and concretization functions are related.

**Definition 2.3. Compatibility.**

An abstraction function \( \alpha \) and concretization function \( \gamma \) are compatible if \( z \in \gamma(\alpha(z)) \) for all \( z \in \mathcal{D}_C \). As an extension, the two functions are strongly compatible if they are compatible and for any \( a \) and \( z \in \gamma(a) \), we have that \( z \notin \gamma(a') \) for any \( a' \neq a \).

**Example 2.1. Predicate domains.** A predicate domain is a well-studied abstract domain induced by a given sequence of predicates \( (p_1, \ldots, p_n) \) about the concrete state. The abstract domain \( \mathcal{D}_A \) consists of \( n \) Boolean variables \( (b_1, \ldots, b_n) \) and so has \( 2^n \) possible elements, one for each valuation to the \( n \) variables. For instance, suppose \( \mathcal{D}_C \) consists of a single integer variable \( x \) whose value is in the range \([-2, 2] \). The single predicate \( x < 0 \) induces an abstract domain with two possible states, respectively representing the concrete states where \( x < 0 \) is true and false. See Figure 1 for a visualization. The abstraction function \( \alpha \) maps each concrete state \( z \) to the abstract state \( (p_1(z), \ldots, p_n(z)) \), and the concretization function \( \gamma \) maps each abstract state \( a \) to the set of concrete states consistent with it: \( \{ z \in \mathcal{D}_C \mid (p_1(z), \ldots, p_n(z)) = a \} \). The functions \( \alpha \) and \( \gamma \) are strongly compatible for predicate domains.

Intuitively, an abstraction represents a set of possible concrete programs, which is formalized as follows:

**Definition 2.4. Concrete semantics of an abstraction.**

The concrete semantics of an abstraction \( \mathcal{A} \), given compatible abstraction and concretization functions \( \alpha \) and \( \gamma \), is a function \( [\mathcal{A}] : \mathcal{D}_C \to 2^{\mathcal{D}_A} \) defined as follows:

\[
[\mathcal{A}](z) = \gamma(\mathcal{A}(\alpha(z))),
\]

where \( \gamma \) is applied to each element of \( \mathcal{A}(\alpha(z)) \).

Ultimately we wish to prove properties about a particular concrete program \( \mathcal{C} \) by reasoning about some simpler \( \mathcal{A} \). From the above definition of an abstraction’s concrete semantics we immediately obtain the following criterion for relating a specific concrete program \( \mathcal{C} \) to \( \mathcal{A} \):
Definition 2.5. Sound over-approximation. Let $A$ be some abstract program with compatible abstraction and concretization functions $\alpha$ and $\gamma$. The tuple $(A, \alpha, \gamma)$ is a sound over-approximation of $C$ if for all $z \in D_C$, $C(z) \in [A](z)$.

In other words, $A$ is sound for $C$ if the result of any concrete execution of $C$ is contained within the possible concretizations of the result of $A$ executed on the abstracted input. Sound over-approximations can be used to verify safety properties of programs, which intuitively express the fact that certain “bad” things never happen (e.g., no null dereferences will occur). Every safety property can be formalized as a requirement that some set $B$ of “bad” states in the concrete program never be reached. To prove that $C(z) \not\in B$ for each concrete state $z$, it suffices to prove that $\gamma([A](a)) \cap B = \emptyset$ for each abstract state $a \in D_A$, where $A$ is a sound over-approximation of $C$.

The semantics above treat programs $C$ and $A$ as black-box input-output functions. Nevertheless, they straightforwardly generalize to assign meaning to every single line of code in the programs, allowing us to establish a sound over-approximation throughout.

2.2 PREDICATE ABSTRACTION

A predicate abstraction is a well-studied program abstraction whose abstract domain is a predicate domain (Graf and Saito, 1997; Ball et al., 2001). See Example 2.1 above for the definition of a predicate domain. Predicate abstractions are known as Boolean programs: the domain $D_A = \{T, F\}^n$. Property checking in Boolean programs is always decidable: a Boolean program has a finite set of states over a fixed number of Boolean variables, making it decidable to obtain the set of reachable states. Given a concrete program $C$ and a set of $n$ predicates $(p_1, \ldots, p_n)$ over the concrete domain $D_C$, the goal of the predicate abstraction process is to construct an abstract Boolean program $A$ that forms a sound over-approximation of $C$.

We use the simple program in Figure 2 as an example to illustrate the predicate abstraction process. The Boolean program induced by the predicates $x<-4$ and $x<3$ is shown in Figure 3. Following the notation of Ball et al. (2001), the $\ast$ operator represents nondeterministic choice, and the Boolean variable associated with predicate $p$ is denoted $\{p\}$. We describe the predicate abstraction process for branches and assignments in turn.

2.2.1 Abstracting Branches

Consider a conditional statement of the form

\[
\text{if (p) \{ \ldots \} else \{ \ldots \}}
\]

in the concrete program. Let $p^T$ denote the strongest propositional formula over the predicates $p_1, \ldots, p_n$ that is implied by $p$ and $p^F$ denote the strongest propositional formula over the predicates $p_1, \ldots, p_n$ that is implied by $\neg p$. These formulas represent the most precise information we can know inside the then and else branches respectively, given the predicates in the abstraction. They can be obtained through queries to an SMT solver, assuming that $p$ and the $n$ predicates are all in decidable logical theories; see Ball et al. (2001) for details. The predicate abstraction process translates the above conditional as follows in the Boolean program:

\[
\text{if (*) \{}
\text{\hspace{1cm} assume(\{p^T\})}
\text{\hspace{1cm} \ldots}
\text{\hspace{1cm} else \{}
\text{\hspace{1.5cm} assume(\{p^F\})}
\text{\hspace{1.5cm} \ldots}
\text{\hspace{1cm} \}}
\]

Here $\{p^T\}$ is $p^T$, but with each predicate $p_i$ replaced by its Boolean counterpart $\{p_i\}$, and similarly for $\{p^F\}$. The statement $\text{assume}(\varphi)$, which is standard in the programming languages community, silently ignores executions which do not satisfy $\varphi$. Note that $\{p^T\}$ and $\{p^F\}$ can simultaneously be true, which allows the execution to nondeterministically take either branch of the conditional.

In the program of Figure 2, we know that $x<0$ is true in the then clause. In Figure 3, the strongest information our abstraction can know at that point is that (the Boolean variable corresponding to) $x<3$ is true. Similarly, $x<0$ is false in the else branch in Figure 2, while the abstraction in Figure 3 only knows that $x<-4$ is false.

```plaintext
1 if(x<0) {
2 \hspace{1cm} x = 0
3 } else {
4 \hspace{1cm} x = x + 1
5 }

Figure 2: A simple concrete program over an integer variable x.

1 if(*) {
2 \hspace{1cm} assume(\{x<3\})
3 {x<-4} = F
4 {x<3} = T
5 } else {
6 \hspace{1cm} assume(\{(!{x<-4})
7 {x<-4} = choose(F, \neg{x<3} \text{ OR } \neg{x<-4})
8 {x<3} = choose(\{x<-4\}, \neg{x<3})
9 }

Figure 3: A predicate abstraction of the program in Figure 2 induced by the predicates $x<-4$ and $x<3$.
```
2.2.2 Abstracting Assignment Statements

Consider an assignment statement of the form $x = e$ in the concrete program. In the corresponding point of the abstract program we must update the values of all Boolean variables to reflect the update to the value of $x$. Suppose we want to update the variable $p_i$. Let $p^F_i$ denote the weakest propositional formula over the predicates $p_1, \ldots, p_n$ such that $p^T_i$ holding before the assignment $x = e$ suffices to ensure that $p_i$ will be true after the assignment. Similarly let $p^F_i$ denote the weakest propositional formula over the predicates $p_1, \ldots, p_n$ such that $p^F_i$ holding before the assignment $x = e$ suffices to ensure that $p_i$ will be false after the assignment. Again an SMT solver can be used to obtain these formulas, leveraging the standard notion of the weakest precondition of program statements with respect to a predicate (Dijkstra 1976). The predicate abstraction process updates the Boolean variable $p_i$ as follows in the Boolean program:

$$p_i = \text{choose}(p^T_i, p^F_i)$$

Here $\text{choose}(\varphi_1, \varphi_2)$ returns $T$ if $\varphi_1$ is satisfied, otherwise returns $F$ if $\varphi_2$ is satisfied, and otherwise chooses nondeterministically between $T$ and $F$.

Consider the assignment statement $x = 0$ in Figure 2. The abstraction process described above will assign $\{x<3\}$ in the Boolean program to $\text{choose}(T, F)$, which simplifies to just $T$ as shown in Figure 3. More interestingly, consider the assignment statement $x = x + 1$ in Figure 2. If $x<4$ is true before the assignment, then we can be sure that $x<3$ is true afterward. If $x<3$ is false before the assignment, then we can be sure that $x<3$ is false afterward. If neither of these is the case, then the abstraction does not have enough information to know the value of $x<3$ after the assignment. Hence in the Boolean program $\{x<3\}$ is assigned to $\text{choose}(\{x<-4\}, \{x<3\})$.

2.2.3 Proving Program Invariants

A predicate abstraction is a sound over-approximation of the original concrete program. Further, because a Boolean program has a finite set of possible states at each point in the program, it can be exhaustively explored via a form of model checking, which conceptually executes the program in all possible ways (Ball and Rajamani 2000). Model checking produces the set of reachable states at each point in the program, and this information can be used to verify invariants of the original program.

Consider the Boolean program in Figure 3. All executions of this program end in a state where the Boolean variable $\{x<-4\}$ has the value $F$. This implies that $x$ always ends in a value greater than or equal to $-4$ in the original program in Figure 2. On the other hand, our predicate abstraction is not precise enough to verify that $x$ always ends in a nonnegative value, though that is true of the original program. A different choice of predicates would enable such reasoning in the abstraction.

3 PROBABILISTIC PROGRAM ABSTRACTION

The primary contribution of this paper is the extension of the non-deterministic program abstractions of the previous section to the probabilistic context. We begin by defining a simple probabilistic programming language. Syntactically, our probabilistic abstractions will simply be probabilistic programs in this language. Next, we generalize the abstraction semantics of Section 2.1 to the probabilistic context, and define soundness criteria for probabilistic program abstractions. Finally, we lift the predicate abstraction process from Section 2.2 to the probabilistic context by placing distributions on the non-deterministic choices. Ultimately this section develops an essential theory for reasoning about program abstractions which are themselves statistical models.

3.1 PROBABILISTIC PROGRAMMING

In probabilistic programming languages, probabilistic semantics are a natural part of the language (Goodman et al. 2008; Carpenter et al. 2016). We define a simple probabilistic programming language, BERN, which contains only (1) Boolean variables; (2) Boolean assignments; (3) if-statements; and (4) a $\text{flip}(\theta)$ operator, which is a Bernoulli random variable with parameter $\theta$. Many existing probabilistic programming languages include within the semantics of the language an $\text{observe}$ statement, which ignores executions that do not satisfy some condition. $\text{observe}$ statements can also be captured by a conditional probability query on the distribution.

An extension to BERN is to introduce a $\text{goto}$ construct, which would allow it to reason about underlying concrete programs with arbitrary control flow. The predicate abstraction framework makes reasoning about loopy concrete programs tractable (Ball et al. 2001); however, we defer lifting the semantics of loopy probabilistic predicate abstractions to future work. In $\text{BERN}$, one could construct a program such as:

```plaintext
Burglary = flip(0.2)
if(Burglary) {
  JohnCalls = flip(0.4)
} else {
  JohnCalls = flip(0.01)
}
observe(JohnCalls)
```
This probabilistic program defines a simple relationship between the event Burglary and the dependent event JohnCalls. This program would allow one to query for Pr(Burglary | JohnCalls).

Probabilistic programming has proven a natural tool for the construction of generative statistical models. As such, infrastructure for computing queries on probabilistic programs has begun to develop in the AI and PL communities (Carpenter et al., 2016; Goodman et al., 2008; Pfeffer, 2009; Wood et al., 2014; Fierens et al., 2013).

### 3.2 Probabilistic Semantics

Section 2.1 identifies both the abstract and concrete semantics of a program abstraction. We generalize these non-deterministic semantics to probabilistic semantics by producing families of compatible probability distributions described by constraints on their support.

Since syntactically abstractions will be probabilistic programs, the abstract semantics of a probabilistic abstraction are simply the semantics of that program, broadly defined.

**Definition 3.1. Abstract semantics.** Let \( a_i, a_o \in \mathcal{D}_A \). The abstract semantics of a probabilistic abstraction \( \mathcal{A} \), denoted \( \Pr_{\mathcal{A}}(a_o \mid a_i) \), is a conditional probability distribution over abstract domain \( \mathcal{D}_A \), which describes the probability of transitioning from an initial set of states \( a_i \) to an output state \( a_o \) under the abstraction \( \mathcal{A} \).

To define the concrete semantics of a probabilistic abstraction, we first need to generalize the concretization function \( \gamma \) to the probabilistic context.

**Definition 3.2. Concretization distribution.** Let \( z \in \mathcal{D}_C \) and \( a \in \mathcal{D}_A \). A concretization distribution is a conditional probability distribution \( \Pr_{\mathcal{A}}(z \mid a) \) that describes the probability of concretizing an abstract state \( a \) to some concrete state \( z \).

In the non-deterministic setting, we were concerned only with membership in the set \( \gamma \). Here, we generalized \( \gamma \) to the probabilistic context by placing a distribution over possible concretizations. Concretization distributions and abstraction functions are related as follows:

**Definition 3.3. Compatibility.** An abstraction function \( \alpha \) and concretization distribution \( \Pr_{\mathcal{A}} \) are compatible when, for all \( z \in \mathcal{D}_C \), \( \Pr_{\mathcal{A}}(z \mid \alpha(z)) > 0 \). Furthermore, these functions are strongly compatible if they are compatible and for any \( a \) and \( z \) such that \( \Pr_{\mathcal{A}}(z \mid a) > 0 \), we have that \( \Pr_{\mathcal{A}}(z \mid a') = 0 \) for all \( a' \neq a \).

We are now in a position to define the concrete semantics of a probabilistic abstraction.

**Definition 3.4. Concrete semantics.** Let \( z_i, z_o \in \mathcal{D}_C \) be some input and output concrete states. The concrete semantics of an abstraction \( \mathcal{A} \), given a compatible abstraction function \( \alpha \) and concretization distribution \( \Pr_{\mathcal{A}} \), is a conditional probability distribution describing the probability of transitioning from \( z_i \) to \( z_o \):

\[
\Pr_{\mathcal{A}}(z_o \mid z_i) = \sum_{a_o \in \mathcal{D}_A} \Pr_{\mathcal{A}}(z_o \mid a_o) \Pr_{\mathcal{A}}(a_o \mid \alpha(z_i)).
\]

In the case when \( \alpha \) and \( \Pr_{\mathcal{A}} \) are strongly compatible, we can refine the above definition:

**Proposition 3.1.** Let \( z_o, z_i \in \mathcal{D}_C \). For strongly compatible \( \alpha \) and \( \Pr_{\mathcal{A}} \), there exists a single \( a_o \) for which \( \Pr_{\mathcal{A}}(z_o \mid a_o) > 0 \). Thus the sum may be collapsed:

\[
\Pr_{\mathcal{A}}(z_o \mid z_i) = \Pr_{\mathcal{A}}(z_o \mid a_o) \Pr_{\mathcal{A}}(a_o \mid \alpha(z_i)).
\]

As an example, we saw previously that predicate domains allow for strongly compatible concretization and abstraction functions. We see in Figure 4 a probabilistic extension to non-deterministic predicate abstraction. The probability of any concrete state (i.e. \( x=-1 \)) is entirely determined by the probability of that concrete state’s sole corresponding abstract state (i.e. \( x<0 \)).

Under the probabilistic semantics, we can define a probabilistic analog of the over-approximation property of \( \mathcal{A} \) as a constraint on the support of \( \Pr_{\mathcal{A}} \).

**Definition 3.5. Sound probabilistic over-approximation.** Let \( \mathcal{A} \) be a probabilistic program abstraction with compatible abstraction function \( \alpha \) and concretization distribution \( \Pr_{\mathcal{A}} \). Then the tuple \( (\mathcal{A}, \alpha, \Pr_{\mathcal{A}}) \) is a sound probabilistic over-approximation of concrete program \( \mathcal{C} \) if for all \( z \in \mathcal{D}_C \), \( \Pr_{\mathcal{A}}(\mathcal{C}(z) \mid z) > 0 \).

#### 3.2.1 Non-Deterministic Semantics

A sound probabilistic over-approximation is a generalization of a sound non-deterministic over-approximation in the sense that it provides a distribution over feasible states. Thus there exists a direct connection between a sound probabilistic over-approximation and a corresponding sound non-deterministic over-approximation by considering states with a non-zero probability, which we explore with the following definitions:

**Definition 3.6. Non-deterministic semantics.** Let \( \mathcal{A} \) be a probabilistic program abstraction with compatible concretization distribution \( \Pr_{\mathcal{A}} \) and abstraction function \( \alpha \). Then there is a corresponding non-deterministic concretization function \( \gamma(\alpha)_i = \{ z \mid \Pr_{\mathcal{A}}(z \mid a) > 0 \} \).
Pr is generated by (1) a distribution over abstract states and (2) one of two concretization distributions: Prγ1 or Prγ2.

and abstract non-deterministic program A(a)↓ = \{a′ | PrA(a′ | a > 0)\}.

We observe that γ(a)↓ is compatible with α if Prγ is compatible with α. The criteria for a sound probabilistic over-approximation are the criteria for a sound over-approximation in the non-deterministic semantics:

**Theorem 3.1. Non-deterministic sound over-approximation.** For any probabilistic program abstraction A with compatible concretization distribution Prγ and abstraction function α, the tuple (A, α, Prγ) is a sound probabilistic over-approximation to concrete program C if and only if the tuple (A(·)↓, α, γ(·)↓) is a sound non-deterministic over-approximation to C.

### 3.2.2 Concretization Invariance

The concrete semantics Pr[LA] are necessary for reasoning about the concrete domain. However, directly analyzing Pr[LA] is made difficult by the necessity of selecting some compatible concretization distribution Prγ. Significantly, in the case when a concrete query can be precisely represented using a set of abstract states, A alone provides sufficient structure to compute a probability in Pr[LA] independent of the choice of Prγ:

**Theorem 3.2. Concretization distribution invariance.** Let A be a probabilistic program abstraction with strongly compatible concretization distribution Prγ and abstraction function α. For any z∈Dγ and a∈DA,

\[
\sum_{z_o \in \gamma(a_o)} Pr[LA](z_o | z_i) = Pr[LA](a_o | α(z_i)).
\]

In other words, the probability of an abstracted event occurring in the concrete semantics is equivalent to the probability of that event in the abstract semantics, regardless of the concretization distribution.

We see a visualization of this theorem in Figure 4. Regardless of whether or not Prγ1 or Prγ2 are chosen,

\[
Pr[LA] (γ(α(x = 1))) = Pr[LA]((-1, -2)) = PrA(\{x < 0\}).
\]

As a consequence, queries performed on the abstraction A represent queries performed on the set of all possible strongly-compatible concretization distributions. Thus, even though in the probabilistic setting we must reason about a distribution over concrete states, we can still lift our analyses to the abstract domain, similar to the benefits of non-deterministic abstraction in Section 2.2.3.

### 3.3 Probabilistic Predicate Abstractions

Thus far we have seen a semantics for a probabilistic program abstraction, but we do not yet have a way to generate one for a particular program. In this section, we seek to generalize predicate abstraction to the probabilistic domain, and show that in general a probabilistic predicate abstraction is a family of Boolean probabilistic programs with Bernoulli flip parameters.

#### 3.3.1 Branch Statements

We saw in Section 2.2.1 that a predicate abstraction of an if-statement is of the form

\[
\text{if(*)} \{\text{assume}(\alpha)\} \text{ else } \{\text{assume}(\beta)\}
\]

where α and β represent the most precise information we can know about the state of predicates at the then and else branches of the program. The behavior of the abstraction is non-deterministic in the case when both α and β hold. A probabilistic predicate abstraction of this statement should explicitly quantify the probability of choosing a particular path when either path is possible in the abstraction.

Consider the predicate abstraction shown in Figure 5. The concrete program is of the form

\[
\text{if}(x<0)\{\ldots}\text{ else }\{\ldots\}. \text{ The corresponding predicate abstraction with the predicates } \{x<3\} \text{ and } \{x<4\} \text{ is of the form}
\]

\[
\text{if(*)} \{\text{assume}(\{x<3\})\} \text{ else } \{\text{assume}(\{x<4\})\}
\]

Intuitively, this means that if the then branch is taken, we know x<3 since x<0; if the else branch is taken, we know x≥4 since x≥0.

We may write an abstract if-statement with a concrete guard γ in an alternative and equivalent way. We note...
that $\gamma \Rightarrow \alpha$ and $\neg \gamma \Rightarrow \beta$. Thus we can remove the assume statements by generating a new guard condition in terms of $\alpha$ and $\beta$, with a non-deterministic $*$ which determines which path to take when $\alpha \land \beta$ holds:

```plaintext
if($\neg \beta \lor (\alpha \land *)$) { ... } else { ... }
```

Extending our running example, we would write the equivalent guard as $if((x<4) \lor ((x<3)\land *))$. This example abstraction is not precise enough to represent the behavior of the concrete program when $x=2$; either path must be permissible in this case for the abstraction to remain an over-approximation of the concrete program.

Thus, a probabilistic predicate abstraction must represent a distribution over paths when $\alpha \land \beta$. Under the semantics of BERN, we may write a probabilistic program which encodes such a distribution:

```plaintext
if($\neg \beta \lor (\alpha \land \text{flip}(\theta))$) { ... } else { ... }
```

Thus a probabilistic predicate abstraction of the running example is $if((x<4) \lor ((x<3)\land \text{flip}(\theta)))$, where $\theta$ represents the conditional probability that the branch is taken given $-4 \leq x < 3$. As long as $0 < \theta < 1$, all concrete executions are contained within the support of this probabilistic program abstraction, implying it is a sound probabilistic over-approximation.

### 3.3.2 Assignment Statements

Section 2.2.2 showed that a concrete assignment is abstracted to a set of predicate assignments of the form $\gamma = \text{choose}(\alpha, \beta)$, where $\gamma$ is a predicate and $\alpha$ and $\beta$ encode the most precise update we can make to $\gamma$. The abstraction behaves non-deterministically: it may assign $\gamma$ to either true or false when $\neg \alpha \land \neg \beta$ holds. Thus, the probabilistic generalization of an assignment statement needs to represent the conditional probability of $\gamma$ given $\neg \alpha \land \neg \beta$.

First, we re-write the $\text{choose}$ statement, introducing a non-deterministic $*$ operator similar to the previous section. We may write an equivalent update to $\gamma$:

$$\gamma = \alpha \lor (\neg \beta \land *)$$

A simple way to represent a probability distribution over the non-deterministic outcomes for this statement in BERN is to replace $*$ with a Bernoulli random variable:

$$\gamma = \alpha \lor (\neg \beta \land \text{flip}(\theta))$$

For example, under this strategy the concrete statement $x=x+1$ with predicates $\{x<3\}$ and $\{x<4\}$ would be abstracted to the BERN program:

$$\{x<4\} = \{x<4\} \land \{x<3\} \land \text{flip}(\theta_1)$$
$$\{x<3\} = \{x<4\} \lor (\{x<3\} \land \text{flip}(\theta_2))$$

We refer to this strategy as independent flip substitution because it models each flip as an independent event.

### 3.4 Predicate Constraints

Multiple predicates that involve the same variable are typically constrained in some way. For example, the predicates $\{(x<3), \{x<4\}\}$ are constrained due to the relationship $\{x<3\} \Rightarrow \{x<4\}$. This constraint is an invariant which increases the precision of the abstraction if it is enforced upon execution of each set of predicate assignments. We call this constraint $I$, and we enforce it by inserting an $\text{observe}(I)$ statement after each set of independent assignments.

#### 3.4.1 Structural Dependence

We present here an alternative to independent flip substitution that obviates the need for predicate constraints.

Consider again the concrete program $x=x-5$. We generate an abstraction using the same predicates as before. However, instead of simply substituting each $*$ for a flip, we condition on the previously assigned value:

```plaintext
\text{if}((x<3)) { \\
    (x<7) = \text{true} \\
} else { \\
    (x<7) = (x<7) \lor \text{flip}(\theta_2)
}
```

---

(a) A simple concrete program over an integer variable $x$.

```plaintext
1  if(x<0) {
2      x = 0
3  } else {
4      x = x + 1
5  }
```

(b) A probabilistic predicate abstraction of Figure 5 with predicates $\{x<3\}, \{x<4\}$ and independent flip substitution.

Figure 5: A concrete program and its associated probabilistic predicate abstraction.
A key point is that, in independent flip substitution, all predicate updates are made simultaneously. Using structural dependence, we update each predicate sequentially, considering all previous decisions. We observe that in this new abstraction, the state \( \{x<3\} \land \neg \{x<7\} \) is guaranteed to have 0 probability via the structure of the abstraction, thus guaranteeing the invariant \( I \) is never violated. Thus, one may sample each flip event independently without risk of generating a 0-probability state which must be ignored, rendering these events independent. In general, one may always construct an abstraction which structurally disallows invalid states.

### 3.4.2 Latent Variable Introduction

It is sometimes the case that two dependent variables can be decoupled by the introduction of a latent variable for which the two dependent variables are conditionally independent, even while subject to constraints \( I \).

Consider the concrete assignment \( x = y + z \). A probabilistic abstraction with predicates \( \{x<0\}, \{x<3\} \) is

\[
\begin{align*}
\{x<0\} &= \text{flip}(\theta_1) \\
\{x<3\} &= \text{flip}(\theta_2)
\end{align*}
\]

Clearly, these two predicates should not be independent; the events \( \{x<0\} \) and \( \{x<3\} \) are determined by a common cause. If we introduce an additional predicate \( \{y+z<1\} \), then the abstraction becomes:

\[
\begin{align*}
\{x<0\} &= \{y+z<1\} \land \text{flip}(\theta_1) \\
\{x<3\} &= \{y+z<1\} \lor \text{flip}(\theta_2)
\end{align*}
\]

The first flip is now the marginal probability of \( \{x<0\} \) given \( \{y+z<1\} \); the second flip is the marginal probability of \( \{x<3\} \) given \( \neg \{y+z<1\} \). The flips have become independent events.

In general, when the predicates are not constrained by an observe statement, we say that the flips are independent.

### 4 Inference

We now briefly turn our attention to the problem of computing \( \Pr_A(a) \), i.e. the probability of the probabilistic predicate abstraction evaluating to some state \( a \).

One option is to treat this as a classic probabilistic program inference problem and rely on an existing probabilistic program inference algorithm, such as the one used found in Stan (Carpenter et al., 2016), Church (Goodman et al., 2008), or Anglican (Wood et al., 2014). However, our particular probabilistic program is over exclusively Boolean variables; we can make many optimizations with this knowledge that these more general probabilistic programming languages can not.

We extend existing techniques from the programming languages literature that are designed for working with Boolean programs to perform inference on BERN. We then use weighted model counting to evaluate queries. We note that abstractions allow one to query the marginal probability of an event at any point in the program, not merely upon program termination.

### 4.1 Probabilistic Model Checking

The problem of computing the set of reachable states in a Boolean program is known as the model checking problem, and has been extensively studied by the programming languages community (Ball and Rajamani, 2000). Commonly one represents the set of reachable states at any point in the program as some Boolean knowledge-base \( \Delta \). In many existing tools, \( \Delta \) is represented using a binary decision diagram. Inference in BERN is thus a simple extension to the traditional model checking paradigm in which we introduce weighted variables for the state of each flip. During model checking, we treat the flip as an unconstrained Boolean variable.

For example, we consider the probabilistic predicate abstraction statement \( \{x<4\} = \{x<4\} \land \text{flip}(\theta) \). We assume \( \Delta = \{x<4\} \land \text{flip}(\theta) \) prior to execution of statement. Following this statement, \( \Delta = ((\{x<4\} \land \text{flip}(\theta)) \land \neg \{y+z<1\}) \lor \{y+z<1\} \). See Ball and Rajamani (2000) for more details.

### 4.2 Weighted Model Counting

Whereas model checking is usually concerned with determining whether \( A \) can reach a particular state, in probabilistic program inference we are concerned with the weighted sum of reachable states, where the weights are induced by the parameters of the flips in each model.

The programming languages community has two primary methodologies for computing the set of reachable states in a Boolean program: (1) knowledge compilation to binary decision diagrams (Ball and Rajamani, 2006), and (2) satisfiability methods (Donaldson et al., 2011). Both of these approaches can be generalized to perform weighted model counting for inference in BERN.

The knowledge compilation approach to model checking is already capable of performing weighted model counting (Darwiche and Marquis, 2001). The satisfiability approach to model checking can be extended to perform weighted model counting. It was shown by Valiant (1979) that this problem is \#P-hard, but a number of recent approximation methods have been explored (Chakraborty et al., 2013; Zhao et al., 2016); see Gomes et al. (2008) for a survey of the subject.
5 LEARNING

Next, we consider the problem of learning the parameters of a probabilistic program abstraction. Here, we will assume that the concrete program $C$ is augmented with a distribution on its input, yielding the concrete program distribution $\Pr[\{1\}](z_0)$. Our goal is for the learned probabilistic abstraction $A$ to approximate this distribution.

In the case when we are not enforcing invariants $I$ through observe statements, the flip parameters are independent and correspond to conditional probabilities. Hence, simple counting derives maximum-likelihood estimates for each parameter of the abstraction, similar to Bayesian network parameter learning.

### Branch Probabilities

Consider a concrete branch statement of the form $\text{if}(\gamma)$, for some concrete expression $\gamma$. We abstract this statement to an $\text{if}$-statement of the form $\text{if}(\neg\beta \lor (\alpha \land \text{flip}(\theta)))$, where $\alpha$ and $\beta$ are abstract expressions. We learn $\theta$ by sampling executions of the concrete program. The maximum likelihood parameter for $\theta$ is the expected number of times that the branch is taken in the concrete program when $\alpha \land \neg\beta$.

### Assignment Probabilities

We consider an assignment $v = \varphi$, for some concrete expression $\varphi$. Its probabilistic predicate abstraction is a set of $i$ simultaneous updates to predicates of the form $\gamma_i = \alpha_i \lor (\neg\beta_i \land \text{flip}(\theta_i))$. If each of these updates is independent, then we can learn the parameters for each $\theta$ independently using a technique identical to that in the previous section. In the case when these events are constrained by an observe statement, the parameters learned by counting are not the maximum likelihood parameters, and need to be estimated using optimization techniques.

In general, the distribution over events in the concrete program and learned abstraction will not match exactly, especially when also enforcing invariants $I$. Consider the concrete program $x = x - 5$. We generate an abstraction to this concrete program with the predicates $\{x<3\}$ and $\{x<7\}$:

- $\{x<3\} = \{x<3\} \lor \{x<7\} \lor \text{flip}(\theta_1)$
- $\{x<7\} = \{x<3\} \lor \{x<7\} \lor \text{flip}(\theta_2)$

If we assume that, prior to the assignment, $x \sim \text{Unif}[0, 15]$, then by counting we would learn $\theta_1 = \Pr(x \in [0, 7] \mid x \geq 7) = 1/8$ and $\theta_2 = \Pr(x \in [0, 11] \mid x \geq 7) = 5/8$. These parameters induce the following joint distribution over the predicates after assignment:

<table>
<thead>
<tr>
<th>${x&lt;3}$</th>
<th>${x&lt;7}$</th>
<th>$\Pr(\cdot \mid x \geq 7)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$(1 - \theta_1)(1 - \theta_2) = 21/64$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$(1 - \theta_1)\theta_2 = 35/64$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\theta_1(1 - \theta_2) = 3/64$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\theta_1\theta_2 = 5/64$</td>
</tr>
</tbody>
</table>

The third row of this table violates our constraint $I = \{x < 3\} \Rightarrow \{x < 7\}$. Thus, the joint distribution over predicates in the concrete program is different from the distribution which we have learned by counting.

### Marginal Matching

Thus far we concluded that (i) concretization invariance allows one to reason about concrete states in $\Pr[A]$ by only looking at the abstract semantics $\Pr[A]$ and (ii) one can generate an abstraction whose independent parameters permit efficient maximum-likelihood learning. In this section we wish to evaluate $\Pr[C]$ by performing a query on a learned probabilistic predicate abstraction $\Pr[A]$. We consider an example where this connection is clear.

For a given concrete query $\Pr[C](\varphi)$, we include $\varphi$ as the sole predicate in a trivial abstraction in which every branch guard is abstracted to a single $\text{flip}$ and every assignment statement is abstracted to an empty statement. On the final line of the abstraction, we assign $\varphi = \text{flip}(\theta)$. It is clear that all flips are independent and the query $\Pr[A](\varphi) = \theta$. According to the learning algorithm presented in the previous sections, the estimation of $\theta$ and subsequent inference on $\Pr[A]$ is exactly performing direct sampling on $C$.

In this example, the abstraction captures the necessary marginal probabilities in order to compute the desired query, since the only marginal exactly corresponds with the query. We generalize this notion to any abstraction with independent flips (i.e., not subject to constraints).

### Proposition 5.1. Independent marginal matching

Let $(A, \alpha, \Pr_\cdot)$ be a probabilistic predicate abstraction with (i) strongly compatible $\Pr_\cdot$ and $\alpha$ and (ii) independent parameters learned from sampling a probabilistic concrete program $\Pr[C]$. Then for any $a_0 \in D_A$,

$$\sum_{z_0 \in \gamma(a_0)_C} \Pr[C](z_0) = \Pr[A](a_0). \quad (1)$$

Consequently, if $A$ is a probabilistic predicate abstraction, then it can be used to compute queries about $\Pr[C]$ involving the predicates from which $A$ is constructed.

6 ILLUSTRATIVE EXPERIMENT

We implemented a probabilistic predicate abstraction engine which uses Z3 (De Moura and Björner, 2008) to construct the probabilistic predicate abstraction, independent flip substitution to create probabilistic predicate assignments, and binary decision diagrams to perform weighted model counting. We use the simple counting-based learning algorithm in Section 5.
Latent Variable Introduction. We illustrate the learning behavior on an example probabilistic program inference problem. Consider the following concrete program:

\[
X = \text{Unif}[0, 15] \\
Y = \text{Unif}[0, 2] \\
Z = 10 \times X + 2 \times Y
\]

We wish to compute \( \Pr(Z < 5 \mid Z < 10) \) upon completion of this program. Due to the independence issues discussed in Section 3.3.2 if we simply use the set of predicates \( \{Z<5\}, \{Z<10\} \) and run the learning algorithm outlined in Section 5 we will model the incorrect joint distribution between the two query predicates. We see that, in Figure 6 with only these predicates the incorrect posterior probability is predicted. However, once we add the predicates \( \{X=0\} \) and \( \{Y<3\} \), the update to \( Z \) becomes an independent event conditioned on the known values of \( X \) and \( Y \), and we can compute the correct posterior probability relating the two predicates.

7 RELATED WORK

Probabilistic reasoning and static analysis. Probabilistic methods have been used in the programming languages and artificial intelligence community to aid in reasoning about deterministic systems. Grigore and Yang (2016) learn a probabilistic model of program behavior in order to guide refinements of a (non-probabilistic) abstraction.

Conversely, static analysis has been used to reason about probabilistic programs. Sankaranarayanan et al. (2013) rely on an interval analysis abstract domain in order to compute bounds on the probability of assertions in a concrete probabilistic program; however, their abstraction itself is not probabilistic. Gehr et al. (2016) use static analysis of a probabilistic program to decompose the problem of inference along paths, which are then dispatched to specialized integration tools depending on the constraints of each path. This work analyzes the original program and does not rely on abstractions.

Probabilistic abstract interpretation. Probabilistic abstract interpretation is used to reason about programs with probabilistic semantics, for example to place upper bounds on the probability mass of a particular path (Monniaux, 2000) or construct Monte-Carlo methods (Monniaux, 2001). This line of work does not explore the connections between abstractions and probabilistic programs, nor does it model concrete program marginals. However, our work does not reason about unbounded loops. The framework of (Coustot and Monerau, 2012) is a highly general framework for reasoning about programs using probabilistic abstract interpretation.

Probabilistic programming systems. Many systems have been developed within the AI and PL communities which tackle the problem of probabilistic program inference, but few utilize abstractions. Systems such as Church (Goodman et al., 2008), Anglican (Wood et al., 2014), Stan (Carpenter et al., 2016), BLOG (Milch et al., 2005), FIGARO (Pfeffer, 2009), and others rely on directly analyzing the concrete program. Weighted model counting and knowledge compilation have been used to perform probabilistic program inference by (Fierens et al., 2013); they do not rely on program abstractions. Approximate inference techniques using knowledge compilation approaches have been extended to the continuous domain for logic programs, but have not been extended to imperative programs (Michels et al., 2016).

8 FUTURE WORK

Probabilistic predicate abstractions – and probabilistic program abstractions in general – are currently unexplored territory for aiding in the analysis of probabilistic programs. Currently, learning the parameters of probabilistic predicate abstractions are limited by strict dependency assumptions, and can not handle concrete programs with loops. The learning method we present requires direct sampling of the concrete program; this currently does not utilize the structure represented by the abstraction. Predicate abstractions naturally generalize to concrete program with loops; probabilistic predicate abstractions could be generalized to this setting as well. Probabilistic predicate abstractions provide promising future avenues for designing new inference and learning procedures for complex deterministic and non-deterministic concrete programs.
References


