THE MATHEMATICS OF CAUSAL RELATIONS

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OUTLINE

• Statistical vs. Causal Modeling: distinction and mental barriers
• Formal semantics for counterfactuals: definition, axioms, graphical representations
• Graphs and Algebra: Symbiosis translation and accomplishments

TRADITIONAL STATISTICAL INFERENCE PARADIGM

Data \[\rightarrow\] \(P\) Joint Distribution \[\rightarrow\] \(Q(P)\) (Aspects of \(P\)) \[\rightarrow\] Inference

e.g.,
Infer whether customers who bought product \(A\) would also buy product \(B\).
\(Q = P(B \mid A)\)

FROM STATISTICAL TO CAUSAL ANALYSIS:
1. THE DIFFERENCES
Probability and statistics deal with static relations

Data \[\rightarrow\] \(P\) Joint Distribution \[\rightarrow\] \(Q(P)\) (Aspects of \(P\)) \[\rightarrow\] Inference

What happens when \(P\) changes?
e.g.,
Infer whether customers who bought product \(A\) would still buy \(A\) if we were to double the price.

FROM STATISTICAL TO CAUSAL ANALYSIS:
1. THE DIFFERENCES (CONT)

1. Causal and statistical concepts do not mix.
   CAUSAL
   Spurious correlation
   Randomization
   Confounding / Effect
   Instrument
   Holding constant
   Explanatory variables
   STATISTICAL
   Regression
   Association / Independence
   “Controlling for” / Conditioning
   Odd and risk ratios
   Collapsibility

2. 

3. 

4. 

FROM STATISTICAL TO CAUSAL ANALYSIS:
1. THE DIFFERENCES

What remains invariant when \(P\) changes say, to satisfy \(P'(price = 2) = 1\)

Data \[\rightarrow\] \(P\) Joint Distribution \[\rightarrow\] \(P'\) Joint Distribution \[\rightarrow\] \(Q(P')\) (Aspects of \(P'\)) \[\rightarrow\] Inference

Note: \(P'(v) \neq P(v \mid price = 2)\)
\(P\) does not tell us how it ought to change
e.g. Curing symptoms vs. curing diseases
e.g. Analogy: mechanical deformation

FROM STATISTICAL TO CAUSAL ANALYSIS: 2. MENTAL BARRIERS

1. Causal and statistical concepts do not mix.
   - **CAUSAL**
     - Spurious correlation
     - Randomization
     - Confounding / Effect
     - Instrument
     - Holding constant
     - Explanatory variables
   - **STATISTICAL**
     - Regression
     - Association / Independence
     - "Controlling for" / Conditioning
     - Odd and risk ratios
     - Collapsibility

2. No causes in – no causes out (Cartwright, 1989)
   - statistical assumptions + data
     \[ \Rightarrow \text{causal conclusions} \]


FROM STATISTICAL TO CAUSAL ANALYSIS: 2. MENTAL BARRIERS

2. No causes in – no causes out (Cartwright, 1989)
   - statistical assumptions + data
     \[ \Rightarrow \text{causal conclusions} \]


4. Non-standard mathematics:
   a) Structural equation models (Wright, 1920; Simon, 1960)
   b) Counterfactuals (Neyman-Rubin, Lewis (x \[ \Rightarrow \] J))

THE STRUCTURAL MODEL PARADIGM

- Data
- Joint Distribution
- Data Generating Model
- \( Q(M) \)
- (Aspects of \( M \))

Inference

\( M \) – Oracle for computing answers to \( Q \)'s.
- e.g.,
  - Infer whether customer \( u \) who bought product \( A \) would still buy \( A \) if we were to double the price.

FAMILIAR CAUSAL MODEL

ORACLE FOR MANIPULATION

\[ X \\
\]
\[ Y \\
\]
\[ Z \\
\]

INPUT

\[ M \]

OUTPUT

STRUCTURAL MODELS AND CAUSAL DIAGRAMS

Definition: A structural causal model is a 4-tuple \( \langle V, U, F, P(u) \rangle \), where
- \( V = \{ V_1,...,V_n \} \) are observable variables
- \( U = \{ U_1,...,U_m \} \) are background variables
- \( F = \{ f_1,...,f_n \} \) are functions determining \( V \),
  \( v_i = f_i(v, u) \)
- \( P(u) \) is a distribution over \( U \)

\( P(u) \) and \( F \) induce a distribution \( P(v) \) over observable variables

Example: Price – Quantity equations in economics

\[ q = b_1p + d_1i + u_1 \]
\[ p = b_2q + d_2w + u_2 \]
Let $X$ be a set of variables in $V$.
The action $do(x)$ sets $X$ to constants $x$ regardless of the factors which previously determined $X$.
$do(x)$ replaces all functions $f_i$ determining $X$ with the constant functions $X=x$, to create a mutilated model $M_x$

$$q = b_1 p + d_1 i + u_1$$
$$p = b_2 q + d_2 w + u_2$$

CAUSAL MODELS AND COUNTERFACTUALS

Definition:
The sentence: "$Y$ would be $y$ (in situation $u$), had $X$ been $x$," denoted $Y_x(u) = y$, means:
The solution for $Y$ in a mutilated model $M_x$ (i.e., the equations for $X$ replaced by $X=x$) with input $U=u$, is equal to $y$.

Joint probabilities of counterfactuals:
$$P(Y_x = y, Z_w = z) = \sum_{u:Y_x(u) = y, Z_w(u) = z} P(u)$$

The super-distribution $P^*$ is derived from $M$.
 Parsimonious, consistent, and transparent

AXIOMS OF CAUSAL COUNTERFACTUALS

$Y$ would be $y$, had $X$ been $x$ (in state $U = u$)

1. Definiteness
   $$\exists x \in X \text{ s.t. } X_y(u) = x$$

2. Uniqueness
   $$(X_y(u) = x) \& (X_y(u) = x') \Rightarrow x = x'$$

3. Effectiveness
   $$X_{XW}(u) = x$$

4. Composition
   $$W_y(u) = w \Rightarrow Y_{XW}(u) = Y_y(u)$$

5. Reversibility
   $$(Y_{XW}(u) = y) \& (W_y(u) = w) \Rightarrow Y_x(u) = y$$

DIFFICULTIES WITH ALGEBRAIC LANGUAGE:

Consider a set of assumptions:
$$Z_x(u) = Z_{yx}(u),$$
$$X_y(u) = X_{zy}(u) = X_z(u) = X(u),$$
$$Y_z(u) = Y_{zx}(u),$$
$$Z_x \perp \{Y_z, X\}$$

Unfriendly:
Consistent?, complete?, redundant?, arguable?

Friendly language:
$$X \rightarrow Z \rightarrow Y$$

GRAPHICAL – COUNTERFACTUALS SYMBIOSIS

Every causal graph expresses counterfactuals assumptions, e.g., $X \rightarrow Y \rightarrow Z$
1. Missing arrows $Y \leftarrow Z$ $Y_{x,z}(u) = Y_y(u)$

2. Missing arcs $Y \rightarrow Z$ $Y_z \perp \{Y_x, Y_y\}$ consistent, and readable from the graph.

Every theorem in SEM is a theorem in Potential Response Model, and conversely.
**DERIVATION IN CAUSAL CALCULUS**

\[ P(c \mid do(s)) = \sum_{t} P(c \mid do(s), do(t)) P(t \mid do(s)) \]

*Probability Axioms*

\[ = \sum_{t} P(c \mid do(s), do(t)) P(t \mid s) \]

*Rule 2*

\[ = \sum_{t} P(c \mid do(t)) P(t \mid s) \]

*Rule 3*

Moreover, \( P(y \mid do(x)) = \sum_z P(y \mid x, z) P(z) \)

("adjusting" for \( Z \))

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**THE BACK-DOOR CRITERION**

Graphical test of identification

\( P(y \mid do(x)) \) is identifiable in \( G \) if there is a set \( Z \) of variables such that \( Z \) \( d \)-separates \( X \) from \( Y \) in \( G_x \).

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**RECENT RESULTS ON IDENTIFICATION**

- \( do \)-calculus is complete
- Complete graphical criterion for identifying causal effects (Shpitser and Pearl, 2006).
- Complete graphical criterion for empirical testability of counterfactuals (Shpitser and Pearl, 2007).

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**STRUCTURAL ANALYSIS: SOME USEFUL RESULTS**

1. Complete formal semantics of counterfactuals
2. Transparent language for expressing assumptions
3. Complete solution to causal-effect identification
4. Legal responsibility (bounds)
5. Non-compliance (universal bounds)
6. Integration of data from diverse sources
7. Direct and Indirect effects,
8. Complete criterion for counterfactual testability

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**DETERMINING THE CAUSES OF EFFECTS (The Attribution Problem)**

- Your Honor! My client (Mr. A) died because he used that drug.

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**DETERMINING THE CAUSES OF EFFECTS (The Attribution Problem)**

- Your Honor! My client (Mr. A) died because he used that drug.

- Court to decide if it is MORE PROBABLE THAN NOT that \( A \) would be alive BUT FOR the drug! \( PN = P(? \mid A \text{ is dead, took the drug}) \geq 0.50 \)
THE PROBLEM

Semantical Problem:
1. What is the meaning of \( PN(x,y) \):
   "Probability that event \( y \) would not have occurred if it were not for event \( x \), given that \( x \) and \( y \) did in fact occur."
   
   •

TYPICAL THEOREMS (Tian and Pearl, 2000)

- Bounds given combined nonexperimental and experimental data
  \[
  \max \left\{ \frac{0}{P(x)} - \frac{P(y|x)}{P(x,y)} \right\} \leq PN \leq \min \left\{ \frac{1}{P(y|x')} - \frac{P(y'|x)}{P(x)} \right\}
  \]
- Identifiability under monotonicity (Combined data)
  \[
  PN = \frac{P(y|x) - P(y'|x)}{P(y|x)} + \frac{P(y|x') - P(y'|x)}{P(x)}
  \]
  corrected Excess-Risk-Ratio

CAN FREQUENCY DATA DECIDE LEGAL RESPONSIBILITY?

<table>
<thead>
<tr>
<th></th>
<th>Experimental</th>
<th>Nonexperimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deaths ( y )</td>
<td>16 14</td>
<td>2 28</td>
</tr>
<tr>
<td>Survivals ( y' )</td>
<td>984 986</td>
<td>998 972</td>
</tr>
</tbody>
</table>

- Nonexperimental data: drug usage predicts longer life
- Experimental data: drug has negligible effect on survival
- Plaintiff: Mr. A is special.
  1. He actually died
  2. He used the drug by choice
- Court to decide (given both data):
  Is it more probable than not that \( A \) would be alive but for the drug?

\[
PN \Delta P(Y_x = y' | x, y) > 0.50
\]
SOLUTION TO THE ATTRIBUTION PROBLEM

• WITH PROBABILITY ONE \( 1 \leq P(y'_{x'} | x, y) \leq 1 \)
• Combined data tell more that each study alone

DIRECT AND INDIRECT EFFECTS

• What is the semantics of direct and indirect effects?

• Can we estimate them from data? Experimental data?

TOTAL, DIRECT, AND INDIRECT EFFECTS HAVE SIMPLE SEMANTICS IN LINEAR MODELS

\[ z = bx + e_1 \]
\[ y = ax + cz + e_2 \]

\[ TE \triangleq \frac{\partial}{\partial x} E(Y | do(x)) = a + bc \]
\[ DE \triangleq \frac{\partial}{\partial x} E(Y | do(x), do(z)) = a \quad Z \text{- independent} \]
\[ IE \triangleq TE - DE = bc \]

CONCLUSIONS

Structural-model semantics, enriched with logic and graphs, provides:
• Complete formal basis for causal reasoning
• Unifies the graphical, potential-outcome and structural equation approaches
• Powerful and friendly causal calculus

WHY DECOMPOSE EFFECTS?

1. Direct (or indirect) effect may be more transportable.
2. Indirect effects may be prevented or controlled.

3. Direct (or indirect) effect may be forbidden

SEMANTICS BECOMES NONTRIVIAL IN NONLINEAR MODELS

(even when the model is completely specified)

\[ z = f(x, e_1) \]
\[ y = g(x, z, e_2) \]

\[ TE \triangleq \frac{\partial}{\partial x} E(Y | do(x)) \]
\[ DE \triangleq \frac{\partial}{\partial x} E(Y | do(x), do(z)) \quad \text{Dependent on } z? \]
\[ IE \triangleq ???? \quad \text{Void of operational meaning?} \]
THE OPERATIONAL MEANING OF DIRECT EFFECTS

\[ \begin{align*}
X & \rightarrow Z \\
& \quad f(x, \varepsilon_1) \\
Y & = g(x, z, \varepsilon_2)
\end{align*} \]

"Natural" Direct Effect of \( X \) on \( Y \):
The expected change in \( Y \) per unit change of \( X \), when we keep \( Z \) constant at whatever value it attains before the change.

\[ E[Y_{x_0Z_{x_0}} - Y_{x_0}] \]

In linear models, \( NDE = \text{Controlled Direct Effect} \)

THE OPERATIONAL MEANING OF INDIRECT EFFECTS

\[ \begin{align*}
X & \rightarrow Z \\
& \quad f(x, \varepsilon_1) \\
Y & = g(x, z, \varepsilon_2)
\end{align*} \]

"Natural" Indirect Effect of \( X \) on \( Y \):
The expected change in \( Y \) when we keep \( X \) constant, say at \( x_0 \), and let \( Z \) change to whatever value it would have under a unit change in \( X \).

\[ E[Y_{x_0Z_{x_0}} - Y_{x_0}] \]

In linear models, \( NIE = TE - DE \)

POLICY IMPLICATIONS

(Who cares?)

What is the direct effect of \( X \) on \( Y \)?
The effect of Gender on Hiring if sex discrimination is eliminated.

GENDER: \[ X \]

QUA LIFICATION: \[ Z \]

IGNORE: \[ Y \]

HIRING

LEGAL DEFINITIONS TAKE THE NATURAL CONCEPTION (FORMALIZING DISCRIMINATION)

"The central question in any employment-discrimination case is whether the employer would have taken the same action had the employee been of different race (age, sex, religion, national origin etc.) and everything else had been the same"

[Carson versus Bethlehem Steel Corp. (70 FEP Cases 921, 7th Cir. (1996))]

\[ \begin{align*}
x & = \text{male, } x' = \text{female} \\
y & = \text{hire, } y' = \text{not hire} \\
z & = \text{applicant's qualifications}
\end{align*} \]

\[ \begin{align*}
\text{NO DIRECT EFFECT} \\
Y'_{X'Z_{x'}} = Y_x, \\
Y'_{XZ_{x'}} = Y_{x'}
\end{align*} \]

SEMANTICS AND IDENTIFICATION OF NESTED COUNTERFACTUALS

Consider the quantity

\[ Q \triangleq E_u[Y_{xZ_{x^*}(u)}(u)] \]

Given \( \langle M, P(u) \rangle \), \( Q \) is well defined

Given \( u, Z_{x^*}(u) \) is the solution for \( Z \) in \( M_{x^*} \), call it \( z \)

\[ Y_{xZ_{x^*}(u)}(u) \] is the solution for \( Y \) in \( M_{xZ_{x^*}} \)

Can \( Q \) be estimated from \{ experimental \ nonexperimental \} data?

GRAPHICAL CONDITION FOR EXPERIMENTAL IDENTIFICATION OF AVERAGE NATURAL DIRECT EFFECTS

Theorem: If there exists a set \( W \) such that

\[ (Y \perp Z \mid W)_{G_{AX}} \text{ and } W \subseteq ND(X \cup Z) \]

\[ NDE(x, x^*; Y) = \sum_{W \subseteq Z} [E(Y_{xZ_{x^*}} \mid w) - E(Y_{x^*Z_{x^*}} \mid w)]P(Z_{x^*} = z \mid w)P(w) \]

Example:
CONCLUSIONS

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- Complete formal basis for causal reasoning
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