THE MATHEMATICS OF CAUSE AND COUNTERFACTUALS

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OUTLINE

- Inference: Statistical vs. Causal distinctions and mental barriers
- Formal semantics for counterfactuals: definition, axioms, graphical representations
- Inference to three types of claims:
  1. Effect of potential interventions
  2. Attribution (Causes of Effects)
  3. Direct and indirect effects

TRADITIONAL STATISTICAL INFERENCE PARADIGM

Data → \( P \) Joint Distribution → \( Q(P) \) (Aspects of \( P \)) → Inference

e.g., Infer whether customers who bought product \( A \) would also buy product \( B \).
\( Q = P(B \mid A) \)

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES

Probability and statistics deal with static relations

Data → \( P \) Joint Distribution → \( P' \) Joint Distribution → \( Q(P') \) (Aspects of \( P' \)) → Inference

What happens when \( P \) changes?
e.g., Infer whether customers who bought product \( A \) would still buy \( A \) if we were to double the price.

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES (CONT)

1. Causal and statistical concepts do not mix.
   CAUSAL
   Spurious correlation
   Randomization / Intervention
   Confounding / Effect
   Instrumental variable
   Strong Exogeneity
   Explanatory variables

   STATISTICAL
   Regression
   Association / Independence
   "Controlling for" / Conditioning
   Odd and risk ratios
   Collapsibility / Granger causality
   Propensity score

2.
3.
4.

FROM STATISTICAL TO CAUSAL ANALYSIS: 1. THE DIFFERENCES

What remains invariant when \( P \) changes say, to satisfy \( P'(\text{price}=2)=1 \)

Data → \( P \) Joint Distribution → \( P' \) Joint Distribution → \( Q(P') \) (Aspects of \( P' \)) → Inference

Note: \( P'(v) = P(v \mid \text{price} = 2) \)
\( P \) does not tell us how it ought to change
e.g. Curing symptoms vs. curing diseases
e.g. Analogy: mechanical deformation
FROM STATISTICAL TO CAUSAL ANALYSIS:
2. MENTAL BARRIERS

1. Causal and statistical concepts do not mix.
   CAUSAL
   Spurious correlation
   Randomization / Intervention
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   Instrumental variable
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   STATISTICAL
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2. No causes in – no causes out (Cartwright, 1989)
   statistical assumptions + data
   causal assumptions
   ⇒ causal conclusions

4. Non-standard mathematics:
   a) Structural equation models (Wright, 1920; Simon, 1960)
   b) Counterfactuals (Neyman-Rubin, Lewis)

WHY CAUSALITY NEEDS
SPECIAL MATHEMATICS

Scientific Equations (e.g., Hooke’s Law) are non-algebraic
e.g., Length (Y) equals a constant (2) times the weight (X)
Correct notation:

\[ Y = 2X \]
\[ X = 1 \]
\[ Y = 2 \]

Process information
The solution

Had X been 3, Y would be 6.
If we raise X to 3, Y would be 6.
Must “wipe out” X = 1.

THE STRUCTURAL MODEL
PARADIGM

Data Distribution
Joint Generating Model
\[ Q(M) \]
(Aspects of M)
Inference

\( M \) – Invariant strategy (mechanism, recipe, law, protocol) by which Nature assigns values to variables in the analysis.

FAMILIAR CAUSAL MODEL
ORACLE FOR MANIPILATION

X

\[ Y \]
\[ Z \]

INPUT

\[ \rightarrow \]

OUTPUT
STRUCTURAL MODELS

Definition: A structural causal model is a 4-tuple \( \langle V, U, F, P(u) \rangle \), where

- \( V = \{ V_1, \ldots, V_n \} \) are observable variables
- \( U = \{ U_1, \ldots, U_m \} \) are background variables
- \( F = \{ f_1, \ldots, f_n \} \) are functions determining \( V \), \( v_i = f(v, u) \)
- \( P(u) \) is a distribution over \( U \)

\( P(u) \) and \( F \) induce a distribution \( P(v) \) over observable variables

STRUCTURAL MODELS AND CAUSAL DIAGRAMS

Example: Price – Quantity equations in economics

\[ p = b_1 p + d_1 q + u_1 \]
\[ q = b_2 q + d_2 w + u_2 \]

STRUCTURAL MODELS AND INTERVENTION

Let \( X \) be a set of variables in \( V \).

The action \( do(x) \) sets \( X \) to constants \( x \) regardless of the factors which previously determined \( X \).

\( do(x) \) replaces all functions \( f \) determining \( X \) with the constant functions \( X = x \), to create a mutilated model \( M_x \).

\[ q = b_1 p + d_1 q + u_1 \]
\[ p = b_2 q + d_2 w + u_2 \]

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CAUSAL MODELS AND COUNTERFACTUALS

Definition:

The sentence: “\( Y \) would be \( y \) (in situation \( u \)), had \( X \) been \( x \),” denoted \( Y(u) = y, X(x) \), means:

The solution for \( Y \) in a mutilated model \( M_x \), (i.e., the equations for \( X \) replaced by \( X = x \)) with input \( U = u \), is equal to \( y \).

The Fundamental Equation of Counterfactuals:

\[ Y_x(u) = Y_{M_x}(u) \]

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The empirical claim of \( Y = f_Y(pay, aty) \)

\[ \text{The empirical claim of } Y_{pay,x}(u) = Y_{M_x}(u) \text{ for } S \text{ outside } PA_Y \]

The empirical claim of \( U_Y \perp U_Z \)

\[ Y_{pay} \perp Z \text{ for } PA_Y \]
CAUSAL MODELS AND COUNTERFACTUALS

Definition:
The sentence: "Y would be y in situation u, had X been x," denoted \( I_u(x) = x \), means:
The solution for \( Y \) in a mutilated model \( M_x \) (i.e., the equations for \( X \) replaced by \( x = x \)) with input \( U = u \), is equal to \( y \).

- Joint probabilities of counterfactuals:
  \[ P(Y_x = y, Z_w = z) = \sum_{u} P(U = u) \]
  In particular:
  \[ P(y | do(x)) = \frac{\sum_{u} P(U = u) | Y_x = y, Z_w = z)}{P(U = u)} \]

AXIOMS OF CAUSAL COUNTERFACTUALS

1. Definiteness
   \( \exists x \in X \text{ s.t. } Y_x(u) = x \)
2. Uniqueness
   \( (Y_x(u) = x) \& (Y_x(u) = x') \Rightarrow x = x' \)
3. Effectiveness
   \( Y_x(u) = x \)
4. Composition
   \( W_x(u) = w \Rightarrow Y_{W_x}(u) = Y_x(u) \)
5. Reversibility
   \( (Y_{XW}(u) = y) \& (W_{XW}(u) = w) \Rightarrow Y_X(u) = y \)

NON-PARAMETRIC STRUCTURAL MODELS

Given \( P(x, y, z) \), should we ban smoking?

Linear Analysis
\[ x = u_1, \]
\[ z = \alpha x + u_2, \]
\[ y = \beta z + \gamma u_1 + u_3. \]
Find: \( \alpha \cdot \beta \)

Nonparametric Analysis
\[ x = f_1(u_1), \]
\[ z = f_2(x, u_2), \]
\[ y = f_3(z, u_1, u_3). \]
Find: \( P(y | do(x)) \)

INFERRING THE EFFECT OF INTERVENTIONS

The problem:
To predict the impact of a proposed intervention using data obtained prior to the intervention.

The solution (conditional):
Causal Assumptions + Data \( \rightarrow \) Policy Claims

1. Mathematical tools for communicating causal assumptions formally and transparently.
2. Deciding (mathematically) whether the assumptions communicated are sufficient for obtaining consistent estimates of the prediction required.
3. Suggesting (if (2) is negative) a set of measurements and experiments that, if performed, would render a consistent estimate feasible.

EFFECT OF INTERVENTION AN EXAMPLE

Given \( P(x, y, z) \), should we ban smoking?

Linear Analysis
\[ x = u_1, \]
\[ z = \alpha x + u_2, \]
\[ y = \beta z + \gamma u_1 + u_3. \]
Find: \( \alpha \cdot \beta \)

Nonparametric Analysis
\[ x = \text{const}, \]
\[ z = f_3(x, u_2), \]
\[ y = f_3(z, u_1, u_3). \]
Find: \( \alpha \cdot \beta \)

Regression vs. Structural Equations (The Confusion of the Century)

Regression (claimless, nonfalsifiable):
\[ Y = ax + \epsilon \]
Structural (empirical, falsifiable):
\[ Y = bx + u_y \]
Assumptions:
\[ E(Y | do(x)) = E(Y | do(x), do(z)) = bx \]

The mothers of all questions:
Q. When would \( b \) equal \( a \)?
A. When all back-door paths are blocked, or \( Y \perp X \)
Q. When is \( b \) estimable by regression methods?
A. Graphical criteria available

The mothers of all questions:
Q. When would \( b \) equal \( a \)?
A. When all back-door paths are blocked, or \( Y \perp X \)
Q. When is \( b \) estimable by regression methods?
A. Graphical criteria available
IDENTIFIABILITY

Definition:
Let $Q(M)$ be any quantity defined on a causal model $M$, and let $A$ be a set of assumptions.

$Q$ is identifiable relative to $A$ iff

$$P(M_1) = P(M_2) \Rightarrow Q(M_1) = Q(M_2)$$

for all $M_1, M_2$ that satisfy $A$.

In other words, $Q$ can be determined uniquely from the probability distribution $P(v)$ of the endogenous variables, $V$, and assumptions $A$.

IDENTIFIABILITY

THE FUNDAMENTAL THEOREM OF CAUSAL INFERENCE

Causal Markov Theorem:
Any distribution generated by Markovian structural model $M$ (recursive, with independent disturbances) can be factorized as

$$P(v_1, v_2, ..., v_n) = \prod_i P(v_i | pa_i)$$

Where $pa_i$ are the (values of) the parents of $V_i$ in the causal diagram associated with $M$.

Corollary-1: (Truncated factorization, Manipulation Theorem)
The distribution generated by an intervention $do(X = x)$ (in a Markovian model $M$) is given by the truncated factorization

$$P(v_1, v_2, ..., v_n | do(x)) = \prod_i P(v_i | pa_i) \mid_{X=x}$$

($G$-estimation)

THE FUNDAMENTAL THEOREM OF CAUSAL INFERENCE

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$$P(v_1, v_2, ..., v_n) = \prod_i P(v_i | pa_i)$$

Where $pa_i$ are the (values of) the parents of $V_i$ in the causal diagram associated with $M$.

Corollary-2: (Parents adjustment formula)
The causal effect of $X$ on $Y$, $P(Y = y | do(X = x))$ (in a Markovian model $M$) is given by

$$P(y \mid do(x)) = \sum_{pa_X} P(y \mid x, pa_X)P(pa_X)$$
EFFECT OF INTERVENTION

THE GENERAL CASE

Find the effect of \( X \) on \( Y \), \( P(y|do(x)) \), given the causal assumptions shown in \( G \), where \( Z_1, \ldots, Z_k \) are auxiliary variables.

\[
\begin{array}{c}
  Z_1 \\
  \downarrow \quad \downarrow \\
  Z_5 \\
  \downarrow \quad \downarrow \\
  X \\
  \downarrow \\
  Y \\
\end{array}
\]

Can \( P(y|do(x)) \) be estimated if only a subset, \( Z \), can be measured?

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ELIMINATING CONFOUNDING BIAS

THE BACK-DOOR CRITERION

\( P(y|do(x)) \) is estimable if there is a set \( Z \) of variables such that \( Z \) \( d \)-separates \( X \) from \( Y \) in \( G_x \).

\[
\begin{array}{c}
  Z_1 \\
  \downarrow \\
  Z_5 \\
  \downarrow \\
  X \\
  \downarrow \\
  Y \\
\end{array}
\]

Moreover, \( P(y|do(x)) = \sum_z P(y|x,z) P(z) \) ("adjusting" for \( Z \))

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INFERENCE ACROSS DESIGNS

Problem:
Predict \( P(y|do(x)) \) from a study in which only \( Z \) can be controlled.

Solution:
Determine if \( P(y|do(x)) \) can be reduced to a mathematical expression involving only \( do(z) \).

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RULES OF CAUSAL CALCULUS

Rule 1: Ignoring observations
\[
P(y|do(x), z, w) = P(y|do(x), w)
\]
if \( (Y \perp \perp Z|X,W)_{G_X} \)

Rule 2: Action/observation exchange
\[
P(y|do(x), do(z), w) = P(y|do(x), z, w)
\]
if \( (Y \perp \perp Z|X,W)_{G_XZ} \)

Rule 3: Ignoring actions
\[
P(y|do(x), do(z), w) = P(y|do(x), w)
\]
if \( (Y \perp \perp Z|X,W)_{G_XZW} \)

---

DERIVATION IN CAUSAL CALCULUS

\[
P(c|do(s)) = \sum_{t} P(c|do(s), do(t)) P(t|do(s))
\]  Probability Axioms

\[
= \sum_{t} P(c|do(s), do(t)) P(t|s)
\]  Rule 2

\[
= \sum_{t} P(c|do(t)) P(t|s)
\]  Rule 3

\[
= \sum_{s'} \sum_{t} P(c|t, s') P(s'|do(t)) P(t|s)
\]  Probability Axioms

\[
= \sum_{s'} \sum_{t} P(c|t, s') P(s'|do(t)) P(t|s)
\]  Rule 2

\[
= \sum_{s'} \sum_{t} P(c|t, s') P(s'|do(t)) P(t|s)
\]  Rule 3

Genotype (Unobserved)

Smoking  Tar  Cancer
**Effect of Intervention Complete Identification**

- Complete calculus for reducing $P(y | do(x), z)$ to expressions void of $do$-operators.
- Complete graphical criterion for identifying causal effects (Shpitser and Pearl, 2006).
- Complete graphical criterion for empirical testability of counterfactuals (Shpitser and Pearl, 2007).

**The Causal Renaissance: Vocabulary in Economics**

From Hoover (2004) "Lost Causes".

**The Causal Renaissance: Useful Results**

1. Complete formal semantics of counterfactuals
2. Transparent language for expressing assumptions
3. Complete solution to causal-effect identification
4. Legal responsibility (bounds)
5. Imperfect experiments (universal bounds for IV)
6. Integration of data from diverse sources
7. Direct and Indirect effects,
8. Complete criterion for counterfactual testability

**Counterfactuals at Work ETT**

1. I took a pill to fall asleep. Should I have?
2. What would terminating a program do to those enrolled?

$ETT = P(Y_x = y | x)$

**ETT - Identification**

Theorem (Shpitser-Pearl, 2009)

$ETT$ is identifiable in $G$ iff $P(y | do(x), w)$ is identifiable in $G'$

Moreover, $ETT = P(y | do(x), w) \mid G'_{w \neq X'}$

**ETT - The Back-Door Criterion**

$P(Y_x = y | x')$ is identifiable in $G$ if there is a set $Z$ of variables such that $Z$ $d$-separates $X$ from $Y$ in $G_x$

Moreover, $P(y | do(x)) = \sum_z P(y | x, z) P(z | x')$ ("adjusting" for $Z$)
Determining the Causes of Effects
(The Attribution Problem)

• Your Honor! My client (Mr. A) died because he used that drug.

• Court to decide if it is more probable than not that Mr. A would be alive but for the drug!
  \[ P_{N} = P(\neg A \mid A \text{ is dead, took the drug}) \geq 0.50 \]

The Problem
Semantical Problem:
1. What is the meaning of \( P_{N}(x,y) \):
   “Probability that event \( y \) would not have occurred if it were not for event \( x \), given that \( x \) and \( y \) did in fact occur.”

   Answer:
   \[ P_{N}(x,y) = P(Y' = y' \mid x, y) \]

   Computable from \( M \)

Analytical Problem:
2. Under what condition can \( P_{N}(x,y) \) be learned from statistical data, i.e., observational, experimental and combined.

Typical Theorems
(Tian and Pearl, 2000)

• Bounds given combined nonexperimental and experimental data
  \[
  \max \left\{ \frac{P(y) - P(y|x)}{P(x,y)} \right\} \leq P_{N} \leq \min \left\{ \frac{1}{P(x,y)} \right\}
  \]

• Identifiability under monotonicity (Combined data)
  \[
  P_{N} = \frac{P(y|x) - P(y|x')} {P(y|x)} + \frac{P(y|x') - P(y|x)} {P(x,y)}
  \]

  Corrected Excess-Risk-Ratio
**CAN FREQUENCY DATA DECIDE LEGAL RESPONSIBILITY?**

<table>
<thead>
<tr>
<th></th>
<th>Experimental</th>
<th>Nonexperimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deaths ($y$)</td>
<td>16</td>
<td>2</td>
</tr>
<tr>
<td>Survivals ($y'$)</td>
<td>984</td>
<td>908</td>
</tr>
</tbody>
</table>

- Nonexperimental data: drug usage predicts longer life
- Experimental data: drug has negligible effect on survival

**Plaintiff:** Mr. A is special.
1. He actually died
2. He used the drug by choice

**Court to decide (given both data):**
Is it more probable than not that $A$ would be alive but for the drug?

$$PN \delta P(Y_{x',y} = y' | x, y) > 0.50$$

**SOLUTION TO THE ATTRIBUTION PROBLEM**

- **WITH PROBABILITY ONE** $1 \leq P(y'_{x',y} | x, y) \leq 1$
- Combined data tell more that each study alone

**EFFECT DECOMPOSITION (direct vs. indirect effects)**

1. Why decompose effects?
2. What is the semantics of direct and indirect effects?
3. What are the policy implications of direct and indirect effects?
4. When can direct and indirect effect be estimated consistently from experimental and nonexperimental data?

**WHY DECOMPOSE EFFECTS?**

1. To understand how Nature works
2. To comply with legal requirements
3. To predict the effects of new type of interventions:
   - Signal routing, rather than variable fixing

**LEGAL IMPLICATIONS OF DIRECT EFFECT**

Can data prove an employer guilty of hiring discrimination?

- **(Gender)** $X$
- **(Qualifications)** $Z$
- **(Hiring)** $Y$

**What is the direct effect of $X$ on $Y$?**

$$E(Y | do(x_1), do(z)) - E(Y | do(x_0), do(z))$$

(averaged over $z$): Adjust for $Z$? No! No!

**NATURAL SEMANTICS OF AVERAGE DIRECT EFFECTS**


Average Direct Effect of $X$ on $Y$: $DE(x_0, x_1; Y)$

The expected change in $Y$, when we change $X$ from $x_0$ to $x_1$ and, for each $u$, we keep $Z$ constant at whatever value it attained before the change.

$$E[Y_{x_1,z|x_0} - Y_{x_0}]$$

In linear models, $DE = \text{Controlled Direct Effect}$
Consider the quantity $Q \Delta E_u [Y_{Zx^*(u)}(u)]$

Given $(M, P(u))$, $Q$ is well defined.

Given $u$, $Zx^*(u)$ is the solution for $Z$ in $M_{x^*}$, call it $z_{x^*}$.

$Y_{Zx^*(u)}(u)$ is the solution for $Y$ in $M_{x^*}$.

Can $Q$ be estimated from experimental or nonexperimental data?

Experimental: nest-free expression

Nonexperimental: subscript-free expression

**Semantics and Identification of Nested Counterfactuals**

**Policy Implications of Indirect Effects**

What is the indirect effect of $X$ on $Y$?

The effect of Gender on Hiring if sex discrimination is eliminated.

**Relations Between Total, Direct, and Indirect Effects**

Theorem 5: The total, direct and indirect effects obey the following equality:

$$TE(x, x^*; Y) = DE(x, x^*; Y) - IE(x^*, x; Y)$$

In words, the total effect (on $Y$) associated with the transition from $x^*$ to $x$ is equal to the difference between the direct effect associated with this transition and the indirect effect associated with the reverse transition, from $x$ to $x^*$.

**Experimental Identification of Average Direct Effects**

Theorem: If there exists a set $W$ such that

$$Y_{x^*} \perp Z_{x^*} \mid W$$

for all $z$ and $x$.

Then the average direct effect

$$DE(x, x^*; Y) = E[Y_{x^*} | w] - E(Y_{x^*})$$

is identifiable from experimental data and is given by

$$DE(x, x^*; Y) = \sum_{w, z} [E(Y_{x^*} | w) - E(Y_{x^*}) | w]P(Z_{x^*} = z | w)P(w)$$

**Graphical Condition for Experimental Identification of Direct Effects**

Theorem: If there exists a set $W$ such that

$$(Y \perp Z \mid W)_{G_{xz}}$$

and $W \subseteq ND(X \cup Z)$

then,

$$DE(x, x^*; Y) = \sum_{w, z} [E(Y_{x^*} | w) - E(Y_{x^*} | w)]P(Z_{x^*} = z | w)P(w)$$

Example:
GENERAL PATH-SPECIFIC EFFECTS (Def.)

Form a new model, $M'_{g*}$, specific to active subgraph $g$

$\psi_{f}(p_{a}, u; g) = f_{i}(p_{a}(g), p_{q}(g), u)$

Definition: $g$-specific effect

$E_{g}(x, x'; Y)_{M} = TE(x, x'; Y)_{M'}_{g}$

Nonidentifiable even in Markovian models

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SUMMARY OF RESULTS

1. Formal semantics of path-specific effects, based on signal blocking, instead of value fixing.
2. Path-analytic techniques extended to nonlinear and nonparametric models.
3. Meaningful (graphical) conditions for estimating direct and indirect effects from experimental and nonexperimental data.

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CONCLUSIONS

Structural-model semantics, enriched with logic and graphs, provides:

- Complete formal basis for causal and counterfactual reasoning
- Unifies the graphical, potential-outcome and structural equation approaches
- Provides friendly and formal solutions to century-old problems and confusions.

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TWO PARADIGMS FOR CAUSAL INFERENCE

Observed: $P(X, Y, Z,...)$

Conclusions needed: $P(Y_{x} = y), P(X_{y} = x | Z = z)...$

How do we connect observables, $X, Y, Z,...$ to counterfactuals $Y_{x}, X_{y}, Z_{y},...$?

**N-R model**

Counterfactuals are primitives, new variables

Super-distribution

$P^{*}(X, Y,..., Y_{x}, Y_{z},...)$

$X, Y, Z$ constrain $Y_{x}, Y_{z},...$

$P(Y_{x} = y) = P_{M_{x}}(Y = y)$

**Structural model**

Counterfactuals are derived quantities

Subscripts modify the model and distribution

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“SUPER” DISTRIBUTION IN N-R MODEL

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$Z$</th>
<th>$Y_{x0}$</th>
<th>$Y_{x1}$</th>
<th>$X_{z0}$</th>
<th>$X_{z1}$</th>
<th>$X_{z0...}$</th>
<th>$U$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0...</td>
<td>$u_{i}$</td>
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<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1...</td>
<td>$u_{2}$</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1...</td>
<td>$u_{3}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0...</td>
<td>$u_{4}$</td>
</tr>
</tbody>
</table>

Inconsistency: $x = 0 \Rightarrow Y_{x0} = Y = xY_{x} + (1-x)Y_{0}$

Defines:

$P^{*}(X, Y, Z,..., Y_{x}, Z_{y},... | X_{z}, Z_{x}, Y_{z},...)$

$P^{*}(Y_{x} = y | Z, X_{z})$

$Y_{x} \perp Z | X_{z}$

---

TYPICAL INFERENCE IN N-R MODEL

Find $P^{*}(Y_{x} = y)$ given covariate $Z$,

$P^{*}(Y_{x} = y) = \sum_{z} P^{*}(Y_{x} = y | z)P(z)$

Assume ignorability:

$Y_{x} \perp X | Z$

$P^{*}(Y_{x} = y | x, z)P(z)$

Assume consistency:

$X_{x} \Rightarrow Y_{x} = y$

$Y_{x} = y | x, z)P(z)$

$P^{*}(y | x, z)P(z)$

Problems:

1) $Y \perp X | Z$ judgmental & opaque
2) Is consistency the only connection between $X, Y$ and $Z$?

Try it: $X \rightarrow Y \rightarrow Z$
DIFFICULTIES WITH ALGEBRAIC LANGUAGE:

Consider a set of assumptions:
\[ Z_X(u) = Z_{XZ}(u), \]
\[ X_Y(u) = X_{YZ}(u) = X_{Z}(u) = X(u), \]
\[ Y_{Z}(u) = Y_{ZX}(u), \]
\[ Z_X \perp \{Y_Z, X\}. \]

Unfriendly:
Consistent?, complete?, redundant?, arguable?
Friendly language:
\[ X \rightarrow Z \rightarrow Y \]

GRAPHICAL – COUNTERFACTUALS

SYMBIOSIS

Every causal graph expresses counterfactuals assumptions, e.g., \( X \rightarrow Y \rightarrow Z \)

1. Missing arrows \( Y \leftarrow Z \quad Y_{XZ}(u) = Y_{X}(u) \)
2. Missing arcs \( Y \quad Z \quad Y_X \perp Z_Y \)

consistent, and readable from the graph.

Every theorem in SCM is a theorem in Potential-Outcome Model, and conversely.

DEMYSTIFYING STRONG IGNORABILITY

\[ \{Y(0), Y(1)\} \perp X | Z \] (SI)
\[ P(y | do(x)) = \sum \frac{P(y | z, x)P(z)}{z} \] (Z-admissible)
\[ (X \perp Y | Z)_{G \setminus X} \] (Back-door)

Is there a \( W \) in \( G \) such that \( (W \perp X | Z)_{0} \Rightarrow \text{SI?} \)

WHAT PROPENSITY SCORE (PS) PRACTITIONERS NEED TO KNOW

\[ L(s) = P(X = 1 | S = s) \quad X \perp S | L(s) \]
\[ \sum_s P(y | s, x)P(s) = \sum_t P(y | t, x)P(t) \]

1. The asymptotic bias of PS is EQUAL to that of ordinary adjustment (for same \( S \)).
2. Including an additional covariate in the analysis CAN SPOIL the bias-reduction potential of others.
3. Choosing sufficient set for PS, if one knows something about the model is a solved problem.
4. That any empirical test of the bias-reduction potential of PS, can only be generalized to cases where the causal relationships among covariates, observed and unobserved is the same.

CONCLUSIONS

Structural Causal Model (SCM), enriched with logic and graphs, provides:

- Complete formal basis for causal and counterfactual reasoning
- Unifies the graphical, potential-outcome and structural equation approaches
- Provides friendly and formal solutions to century-old problems and confusions.