

Confounding Equivalence in Observational Studies (or, when are two measurements equally valuable for effect estimation?)

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Abstract

The paper provides a simple test for deciding, from a given causal diagram, whether two sets of covariates have the same bias-reducing potential under adjustment. The test requires that one of the following two conditions holds: either (1) both sets are admissible (i.e., satisfy the back-door criterion) or (2) the Markov boundaries surrounding the treatment variable(s) are identical in both sets. Applications to covariate selection and model testing are discussed.

1 Introduction

The common method of reducing confounding bias in observational studies is to *adjust* for a set of covariates judged to be “confounders,” that is, variables capable of producing spurious associations between treatment and outcome, not attributable to their causal dependence. It is now a common understanding that the bias-reducing potential of any set of covariates depends crucially on the causal relationships among all variables affecting treatment or outcome, hidden as well as visible, and that such relationships can effectively be represented in the form of directed acyclic graphs (DAG’s) (Pearl, 1995; Lauritzen, 2001; Spirtes et al., 2000; Glymour and Greenland, 2008).

Most studies of covariate selection have aimed to define and identify “admissible” sets of covariates, also called “sufficient sets,” namely, a set of covariates that, if adjusted for,

would yield asymptotically unbiased estimates of the causal effect of interest (Stone, 1993; Greenland et al., 1999; Pearl, 2000). A graphical criterion for selecting an admissible set is given by the “back-door” test (Pearl, 1993, 2000) which was shown to entail zero-bias, or “no confoundedness,” assuming correctness of the causal assumptions encoded in the DAG. Related notions are “exchangeability” (Greenland and Robins, 1986), “exogeneity” (Engle et al., 1983), and “strong ignorability” (Rosenbaum and Rubin, 1983).

This paper addresses a different question: Given two sets of covariates in a DAG, decide if the two are equally valuable for adjustment, namely, whether adjustment for one set is guaranteed to yield the same asymptotic bias as adjustment for the other.

The reasons for posing this question are several. First, an investigator may wish to assess, prior to taking any measurement, whether two candidate sets of covariates, differing substantially in dimensionality, measurement error, cost, or sample variability are equally valuable in their bias-reduction potential. Second, assuming that the structure of the underlying DAG is only partially known, one may wish to assess, using c -equivalence tests, whether a given structure is compatible with the data at hand; structures that predict equality of post-adjustment associations must be rejected if, after adjustment, such equality is not found in the data. Finally, tests for bias equality offer computational advantages over tests for conditional independencies; characterizing graphs in term of the former, may lead to more effective ways of inferring graph structures from data.

2 Preliminaries: c -equivalence and admissibility

Let X, Y , and Z be three disjoint subsets of discrete variables, and $P(x, y, z)$ their joint distribution. We are concerned with expressions of the type

$$A(x, y, Z) = \sum_z P(y|x, z)P(z) \quad (1)$$

Such expressions, which we name “adjustment estimands,” are often used to approximate the causal effect of X on Y , where the set Z is chosen to include variables judged to be “confounders.” By *adjusting* for these variables, one hopes to create conditions that eliminate spurious dependence and thus obtain an unbiased estimate of the causal effect of X and Y , written $P(y|do(x))$ (see Pearl (2000) for formal definition and methods of estimation).

Definition 1. (c -equivalence)

Define two sets, T and Z as c -equivalent (relative to X and Y), written $T \approx Z$, if the following equality holds for every x and y :

$$\sum_t P(y|x, t)P(t) = \sum_z P(y|x, z)P(z) \quad \forall x, y \quad (2)$$

or

$$A(x, y, T) = A(x, y, Z) \quad \forall x, y$$

This equality guarantees that, if adjusted for, sets T and Z would produce the same asymptotic bias relative to the target quantity.

Definition 2. (*Admissibility*)

Let $P(y|do(x))$ stand for the “causal-effect” of X on Y , i.e., the distribution of Y after setting variable X to a constant $X = x$ by external intervention. A set Z of covariates is said to be “admissible” (for adjustment) relative to the causal effect of X on Y , if the following equality holds:

$$\sum_z P(y|x, z)P(z) = P(y|do(x)) \quad (3)$$

Equivalently, one can define admissibility using the equalities:

$$P(y|do(x)) = P(Y_x = y) = \sum_u P(y|x, u)P(u) \quad (4)$$

where Y_x is the counterfactual or “potential outcome” variable (Neyman, 1923; Rubin, 1974) and U ranges over all variables that affect either X or Y , hidden as well as observables (Stone, 1993). The equivalence of the three definitions is demonstrated in Pearl (2000).¹

Definition 3. (*d-separation*)

A set S of nodes in a graph G is said to block a path p if either (i) p contains at least one arrow-emitting node that is in S , or (ii) p contains at least one collision node that is outside S and has no descendant in S . If S blocks all paths from X to Y , it is said to “d-separate X and Y ,” and then, X and Y are independent given S , written $X \perp\!\!\!\perp Y|S$, in every probability distribution that can be generated by a process structured along G (Pearl, 1988).

Lemma 1. (*The back-door criterion*)

Let G be a directed acyclic graph (DAG) that encodes the causal relationships between variables in a problem, observables as well as unobservable. A sufficient condition for a subset S of covariates to be admissible is that it satisfies the following two conditions (Pearl, 1993):

1. No element of S is a descendant of X
2. The elements of S “block” all “back-door” paths from X to Y , namely all paths that end with an arrow pointing to X .

For proof and intuition behind the back-door test, especially the requirement of no descendants, see (Pearl, 2009, p. 339).

Clearly, if two subsets Z and T are admissible they must be c -equivalent, for their adjustment estimands coincide with the causal effect $P(y|do(x))$. Therefore, a trivial graphical condition for c -equivalence is for Z and T to satisfy the back-door criterion of Lemma 1. This condition, as we shall see in the next section, is rather weak; c -equivalence extends beyond admissible sets.

¹The $do(x)$ notation is preferred here because it rests on realizable variables and explicit causal processes. The expression $\sum_u P(y|x, u)P(u)$ has the appearance of a standard statistical parameter, but is in fact a causal notion, for it assumes knowledge of “all causes” and cannot be defined in terms of distributions of observed variables (Pearl 2000, Chapter 6; 1998).

3 Conditions for c -equivalence

Theorem 1. *A sufficient condition for the c -equivalence of T and Z is that Z satisfies:*

$$\begin{array}{ll} (X \perp\!\!\!\perp Z|T) & (i) \\ (Y \perp\!\!\!\perp T|X, Z) & (ii) \end{array}$$

Proof:

Conditioning on Z , (ii) permits us to rewrite the left-hand side of (2) as

$$\begin{aligned} A(x, y, T) &= \sum_t P(t) \sum_z P(y|z, x, t) P(z|t, x) \\ &= \sum_t P(t) \sum_z P(y|z, x) P(z|t, x) \end{aligned}$$

and (i) further yields $P(z|t, x) = P(z|t)$, from which (2) follows:

$$\begin{aligned} A(x, y, T) &= \sum_t \sum_z P(y|z, x) P(z, t) \\ &= \sum_z P(y|z, x) P(z) \\ &= A(x, y, Z) \end{aligned}$$

Corollary 1. *A sufficient condition for the c -equivalence of T and Z is that either one of the following two conditions holds:*

$$\begin{array}{ll} C^* : & X \perp\!\!\!\perp Z|T \quad \text{and} \quad Y \perp\!\!\!\perp T|Z, X \\ C^{**} : & X \perp\!\!\!\perp T|Z \quad \text{and} \quad Y \perp\!\!\!\perp Z|T, X \end{array} \quad (5)$$

Proof:

C^* permits us to derive the right hand side of Eq. (2) from the left hand side, while C^{**} permits us to go the other way around.

The conditions offered by Theorem 1 and Corollary 1 do not characterize *all* equivalent pairs, T and Z . For example, consider the graph in Fig. 1, in which each of $T = \{V_1, W_2\}$ and

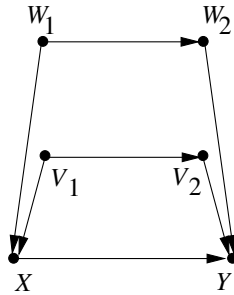


Figure 1: The sets $T = \{V_1, W_1\}$ and $Z = \{V_2, W_2\}$ satisfy the conditions of Theorem 1. The sets $T = \{V_1, W_2\}$ and $Z = \{V_2, W_1\}$ block all back-door paths between X and Y , hence they are admissible and c -equivalent. Still they do not satisfy the conditions of Theorem 1.

$Z = \{V_2, W_1\}$ is admissible (by virtue of satisfying the back-door criterion); they must therefore be c -equivalent. Yet neither C^* nor C^{**} holds in this case.

On the other hand, condition C^* can detect the c -equivalence of some non-admissible sets, such as $T = \{W_1\}$ and $Z = \{W_1, W_2\}$. These two sets are non-admissible for they fail to block the back-door path $X \leftarrow V_1 \rightarrow V_2 \rightarrow Y$, yet they are c -equivalent according to Theorem 1; (i) is satisfied by d -separation, while (ii) is satisfied by subsumption ($T \subseteq Z$).

It is interesting to note however that $Z = \{W_1, W_2\}$, while c -equivalent to $\{W_1\}$, is not c -equivalent to $T = \{W_2\}$, though the two sets block the same paths in the graph.² Indeed, this pair does not meet the test of Theorem 1; choosing $T = \{W_2\}$ and $Z = \{W_1, W_2\}$ violates condition (i) since X is not d -separated from W_1 , while choosing $Z = \{W_2\}$ and $T = \{W_1, W_2\}$ violates condition (ii) by unblocking the path $W_1 \rightarrow X \leftarrow V_1 \rightarrow V_2 \rightarrow Y$. Likewise, the sets $T = \{W_1\}$ and $Z = \{W_2\}$ block the same path and, yet, are not c -equivalent; they fail indeed to satisfy condition (ii) of Theorem 1.

We are now ready to broaden the scope of Theorem 1 and derive a condition (Theorem 2 below) that detects *all* c -equivalent subsets in a graph.

Definition 4. (*Markov boundary*)

For any set of variables S , let S_m be the minimal subset of S that satisfies the condition

$$X \perp\!\!\!\perp S | S_m \quad (6)$$

In other words, measurement of S_m renders X independent of all other members of S , and no proper subset of S_m has this property. This minimal subset is called the *Markov Boundary* of X relative to S , or “Markov boundary” for short.

The Markov boundary as defined above is known to be unique for all strictly positive distributions (Pearl, 1988, p. 97). We can also define this notion relative to a graph G by interpreting the $\perp\!\!\!\perp$ sign in Eq. (6) to denote d -separation. We will not distinguish between the graphical and statistical definitions because all our theorems would be valid in both interpretations.³

Lemma 2. Every set of variables, S , is c -equivalent to its Markov boundary S_m .

Proof.

Choosing $Z = S$ and $T = S_m$ satisfies the two conditions of Theorem 1; (i) is satisfied by the definition of S_m , while (ii) is satisfied by subsumption ($T \subseteq Z$).

Theorem 2. Let Z and T be two sets of variables containing no descendant of X . A necessary and sufficient conditions for Z and T to be c -equivalent is that at least one of the following conditions holds:

1. $Z_m = T_m$

or

2. Z and T are admissible

²The reason is that the strength of the association between X and Y , conditioned on W_2 , depends on whether we also condition on W_1 . Else, $P(y|x, w_2)$ would be equal to $P(y|x, w_1, w_2)$ which would render Y and W_1 independent given X and W_2 . But this is true only if the path (X, V_1, V_2, Y) is blocked.

³Note however that the graphical definition gives rise to a *unique* minimal subset S_m that satisfies Eq. (6). This follows from the intersection property of d -separation (Pearl, 1988, p. 97).

Proof

1. Proof of sufficiency:

Condition 2 is sufficient since admissibility renders the two adjustment estimands in (2) equal to the causal effect. Condition 1 is sufficient by reason of Lemma 2, which yields:

$$Z \approx Z_m \approx T_m \approx T$$

2. Proof of necessity:

If exactly one of (Z, T) is admissible then Z and T are surely not c -equivalent, for their adjustment estimands would differ for some parameterization of the graph. Assume that both Z and T are not admissible or, equivalently, that none of Z_m or A_m is admissible. Then there is a back-door path p from X to Y that is not blocked by either Z_m or T_m . If, in addition, condition (1) is violated (i.e., Z_m differs from T_m) then, excluding the degenerate case where T_m and Z_m are both disconnected from X , there must be a path p_1 from either Z_m to X that is not blocked by T_m or a path p_2 from T_m to X that is not blocked by Z_m . Assuming the former case, we conclude that if (1) and (2) are both violated, there must be an unblocked path p_1 from Z_m to X followed by a back door path p from X to Y . The existence of this path implies that for some values z' and z'' of Z and for all t of T we have

$$\begin{aligned} P(y|z', t, x) &\neq P(y|t, x) \\ P(z''|x, t) &\neq P(z''|t) \end{aligned}$$

Thus, conditional on t , Z acts as an instrumental variable with respect to the pair (X, Y) and, based on the examples in Wooldridge (2009), there is a parametrization of the graph that would yield the inequality $A(x, y, T_m) \neq A(x, y, Z_m)$. This proves the necessary part of Theorem 2.

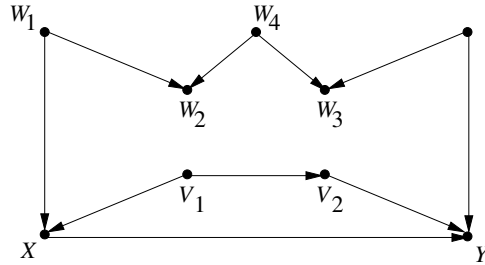


Figure 2: W_3 and W_4 are non-admissible yet c -equivalent; both having $\emptyset$ as a Markov boundary. However, W_2 and W_3 are not c -equivalent with Markov boundaries W_2 and $\emptyset$, respectively.

Figure 2 illustrates the power of Theorem 2. In this model, no subset of $\{W_1, W_2, W_3\}$ is admissible (because of the back-door path through V_1 and V_2) and, therefore, equality of Markov boundaries is necessary and sufficient for c -equivalence among any two such subsets. Accordingly, we can conclude that $T = \{W_1, W_2\}$ is c -equivalent to $Z = \{W_1, W_3\}$, since $T_m = W_1$ and $Z_m = W_1$. Note that W_1 and W_2 , though they block the same set of paths, are not

c -equivalent since $T_m = W_1 \neq Z_m = W_2$. Indeed, each of W_1 and W_2 is an instrumental variable relative to $\{X, Y\}$, with potentially different strengths, hence potentially different adjustment estimands. Sets W_4 and W_3 however are c -equivalent, because the Markov boundary of each is the null set, $\{\emptyset\}$.

At this point it is worth noting that, based on Theorem 2, c -equivalence can be tested by a step-wise process,

$$T \approx T_1 \approx T_2 \approx \dots \approx Z$$

where each intermediate set is obtained from its predecessor by an addition or deletion of one variable only. This can be seen by organizing the chain into three sections.

$$T \approx \dots \approx T_m \approx \dots \approx Z_m \approx \dots \approx Z$$

The transition from T to T_m entails the deletion from T of all nodes that are not in T_m ; one at a time, in any order. Similarly, the transition from Z_m to Z builds up the full set Z from its Markov boundary Z_m ; again, in any order. Finally, the middle section, from T_m to Z_m , amounts to traversing a chain of admissible sets, using both deletion and addition of nodes, one at a time. A Theorem due to Tian et al. (1998) ensures that such a step-wise transition is always possible between any two admissible sets. In case T or Z are non-admissible, the middle section must degenerate into an equality $T_m = Z_m$, or else, c -equivalence does not hold.

Figure 2 can be used to illustrate this stepwise transition from $T = \{W_1, W_2, V_1\}$ to $Z = \{V_2, W_3\}$. Starting with T , we obtain:

$$T = \{W_1, W_2, V_1\} \approx \{W_2, V_1\} \approx \{V_1\} \approx \{V_1, V_2\} \approx \{V_2\} \approx \{V_2, W_3\} = Z$$

If, however we were to attempt a stepwise transition between $T = \{W_1, W_2, V_1\}$ and $Z = \{W_3\}$, we would obtain:

$$T = \{W_1, W_2, V_1\} \approx \{W_2, V_1\} \approx \{V_1\}$$

and would be unable to proceed toward $Z_m = \{W_3\}$. The reason lies in the non-admissibility of Z which necessitates the equality $T_m = Z_m$, contrary to Markov boundaries shown in the graph.

Note also that each step in the process $T \approx \dots \approx T_m$ (as well as $Z_m \approx \dots \approx Z$) is licensed by condition (i) of Theorem 1, while each step in the intermediate process $T_m \approx \dots \approx Z_m$ is licensed by condition (ii). Both conditions are purely statistical and do not invoke the causal reading of “admissibility.” This means that condition 2 of Theorem 2 may be replaced by the requirement that Z and T satisfy the back-door test in any diagram compatible⁴ with $P(x, y, z, t)$; the direction of arrows in the diagram need not convey causal information. Further clarification of the statistical implications of the admissibility condition, is given in the next section.

4 Discussion and refinements

Theorem 2, while providing a necessary and sufficient condition for c -equivalence, raises an interesting theoretical question. Admissibility is a causal notion, (i.e., resting on causal assumptions about the direction of the arrows in the diagram) while c -equivalence is purely

⁴A diagram G is said to be compatible with a probability function P if G can generate P , i.e., if every d -separation in G corresponds to a valid conditional independence in P .

statistical. Why need one resort to causal assumptions to characterize a property that relies on no such assumption? Evidently, the notion of admissibility as it was used in the proof of Theorem 2 was merely a surrogate carrier of statistical information; its causal reading was irrelevant. The question then is whether Theorem 2 could be articulated using purely statistical conditions, avoiding admissibility altogether, as is done in Theorem 1.

We will show that the answer is positive; Theorem 2 can be rephrased using a statistical test for c -equivalence. It should be noted though, that the quest for statistical characterization is of merely theoretical interest; rarely is one in possession of prior information about conditional independencies, (as required by Theorem 1), that is not resting on causal knowledge (of the kind required by Theorem 2). The utility of statistical characterization surfaces when we wish to confirm or reject the structure of the diagram. We will see that the statistical reading of Theorem 2 has testable implication that, if failed to fit the data, may help one select among competing graph structures.

Our first step is to apply Theorem 1 to the special case where T is a subset of Z .

Theorem 3. (*Set-subset equivalence*)

Let T and S be two disjoint sets. A sufficient condition for the c -equivalence of T and $Z = T \cup S$ is that S can be partitioned into two subsets, S_1 and S_2 , such that:

$$(i') \quad S_1 \perp\!\!\!\perp X|T$$

and

$$(ii') \quad S_2 \perp\!\!\!\perp Y|S_1, X, T$$

Proof:

Starting with

$$A(x, y, T \cup S) = \sum_t \sum_{s_1} \sum_{s_2} P(y|x, t, s_1, s_2) P(s_1, s_2, t)$$

(ii') permits us to remove s_2 from the first factor and write

$$\begin{aligned} A(x, y, T \cup S) &= \sum_t \sum_{s_1} \sum_{s_2} P(y|x, t, s_1) P(s_1, s_2, t) \\ &= \sum_t \sum_{s_1} P(y|x, t, s_1) P(s_1, t) \end{aligned}$$

while (i') permits us to reach the same expression from $A(x, y, T)$:

$$\begin{aligned} A(x, y, T) &= \sum_t \sum_{s_1} P(y|x, t, s_1) P(s_1|x, t) P(t) \\ &= \sum_t \sum_{s_1} P(y|x, t, s_1) P(s_1, t) \end{aligned}$$

which proves the theorem.

Theorem 3 generalizes similar theorems by Stone (1993) and Robins (1997), in which $T \cup S$ is assumed to be admissible (see also (Greenland et al., 1999)). The importance of this generalization was demonstrated by several examples in Section 3. Theorem 3 on the other

hand invokes only the distribution $P(x, y, z, t)$ and makes no reference to $P(y|do(x))$ or to the “all-causes” set U , as in Eq. (3) or (4).

Theorem 3 can also be proven by double application of Theorem 1; first showing the c -equivalence of T and $\{T \cup S_1\}$ using (i) (with (ii) satisfied by subsumption), then showing the c -equivalence of $\{T \cup S_1\}$ and $\{T \cup S_1 \cup S_2\}$ using (ii) (with (i) satisfied by subsumption).

The advantage of Theorem 3 over Theorem 1 is that it allows certain cases of c -equivalence to be verified in a single step. In Fig. 1, for example, both (i') and (i'') are satisfied for $T = \{V_1, W_2\}$, $S_1 = \{V_2\}$, and $S_2 = \{W_1\}$. Therefore, $T = \{V_1, W_2\}$ is c -equivalent to $\{T \cup S\} = \{V_1, V_2, W_1, W_2\}$

The weakness of Theorem 3 is that it is applicable to set-subset relations only. A natural attempt to generalize the theorem would be to posit the condition that T and Z each be c -equivalent to $T \cup Z$, and use Theorem 3 to establish the required set-subset equivalence. While perfectly valid, this condition is still not complete; there are cases where T and Z are c -equivalent, yet none is c -equivalent to their union. For example, consider the path

$$X \rightarrow T \leftarrow L \rightarrow Z \leftarrow Y$$

Each of T and Z leaves the path between X and Y blocked, which renders them c -equivalent, yet $\{T \cup Z\}$ unblocks that path. Hence, $T \approx Z$ and $T \not\approx T \cup Z$. This implies that sets T and Z would fail the proposed test, even though they are c -equivalent.

The remedy can be obtained by re-invoking the notion of Markov boundary (Definition 4) and Lemma 2.

Theorem 4. *Let T and Z be two sets of covariates, containing no descendant of X and let T_m and Z_m be their Markov boundaries. A necessary and sufficient condition for the c -equivalence of T and Z is that each of T_m and Z_m be c -equivalent to $T_m \cup Z_m$ according to the set-subset criterion of Theorem 3.*

Proof

1. Proof of sufficiency:

If T_m and Z_m are each c -equivalent to $T_m \cup Z_m$ then, obviously, they are c -equivalent themselves and, since each is c -equivalent to its parent set (by Lemma 2) T and Z are c -equivalent as well.

2. Proof of necessity:

We need to show that if either T_m or Z_m is not c -equivalent to their union (by the test of Theorem 3), then they are not c -equivalent to each other. We will show that using “admissibility” as an auxiliary tool. We will show that failure of $Z_m \approx T_m \cup Z_m$ implies non-admissibility, and this, by the necessary part of Theorem 2, negates the possibility of c -equivalence between Z and T . The proof relies on the monotonicity of d -separation over minimal subsets (Appendix 1), which states that, for any graph G , and any two subsets of nodes T and Z , we have:

$$(X \perp\!\!\!\perp Y | Z_m)_G \quad \& \quad (X \perp\!\!\!\perp Y | T_m)_G \Rightarrow (X \perp\!\!\!\perp Y | Z_m \cup T_m)_G$$

Applying this to the subgraph consisting of all back-door paths from X to Y , we conclude that admissibility is preserved under union of minimal sets. Therefore, the admissibility of

Z_m and T_m (hence of Z and T) entails admissibility of $Z_m \cup T_m$. Applying Theorem 2, this implies the necessity part of Theorem 3.

Theorem 4 unveils the statistical implications of the admissibility requirement in Theorem 2. Admissibility ensures the two c -equivalence conditions:

$$T_m \approx \{T_m \cup Z_m\} \quad (7)$$

$$Z_m \approx \{T_m \cup Z_m\} \quad (8)$$

In other words, given any DAG G compatible with the conditional independencies of $P(x, y, t, z)$, whenever Z and T are admissible in G , we should be able to prove Eqs. (7) and (8) using the set-subset conditions of Theorem 3. Applying these conditions yields two alternative conditions which subsume $T_m = Z_m$ and permit us to reformulate Theorem 2 in purely probabilistic terms, clear of the graphical notion, of admissibility and back-door. We explicate this formulation in a Theorem.

Theorem 5. *A necessary and sufficient conditions for Z and T to be c -equivalent is that the following conditions both hold:*

$$\{T_m \cup Z_m\}_m \perp\!\!\!\perp Y | X, Z_m \quad (9)$$

$$\{T_m \cup Z_m\}_m \perp\!\!\!\perp Y | X, T_m \quad (10)$$

Proof:

To establish $Z_m \approx \{Z_m \cup T_m\}$ it is necessary and sufficient to establish

$$Z_m \approx \{Z_m \cup T_m\}_m \triangleq U_m$$

Let us establish the equivalence $Z_m \approx U_m$ in two steps each involving set-subset relations. First, establish $U_m \approx \{U_m \cup Z_m\}$ then $\{U_m \cup Z_m\} \approx Z_m$.

The first equivalence is satisfied automatically, with no added prerequisites; being a Markov boundary of $\{T_m \cup Z_m\}$, U_m must also be a Markov boundary of any subset of $\{T_m \cup Z_m\}$ that includes U_m . ($(X \perp\!\!\!\perp S | Z) \implies (X \perp\!\!\!\perp S' | Z)$ for any subset S' of S .)

Now let us examine what conditions are required to establish the second c -equivalence,

$$(U_m \cup Z_m) \approx Z_m$$

The left hand side can be partitioned into two disjoint sets Z_m and $\{U_m \cup Z_m\} \setminus Z_m = U_m \setminus Z_m$. Our task then is to establish

$$(U_m \setminus Z_m \cup Z_m) \approx Z_m$$

According to Theorem 3 this c -equivalence holds iff $U_m \setminus Z_m$ can be partitioned into two parts, T_1 and T_2 , satisfying

$$(i') T_1 \perp\!\!\!\perp X | Z_m \quad \text{and} \quad (ii') T_2 \perp\!\!\!\perp Y | X, T_1, Z_m$$

We will show that $T_2 = U_m \setminus Z_m, T_1 = \emptyset$ is the only partition that can satisfy (i') and (ii'). Indeed, since $U_m \setminus Z_m$ is contained in the Markov boundary of a set (e.g., $\{T_m \cup Z_m\}$), no member

of it can be separated from X by a subset of that set (like Z_m). (We are assuming intersection here). Thus, choosing $T_1 = 0$, yields the condition:

$$U_m \setminus Z_m \perp\!\!\!\perp Y | X, Z_m$$

with a symmetrical condition for T_m . However, since one can always add part of the conditioning set (Z_m) to the left hand side of a conditional independence statement, we get

$$U_m \perp\!\!\!\perp Y | X, Z_m$$

with a symmetrical condition for T_m :

$$U_m \perp\!\!\!\perp Y | X, T_m$$

This completes the proof. $\square$

Theorem 5 explicates the statistical role that admissibility plays in the definition of c -equivalence. It further translates the statistical notion of c -equivalence into the language of conditional independence and provides us, therefore with another graphical test of characterizing the former.

We illustrate these conditions using Fig. 2. Taking $T = \{W_2, V_2\}$ and $Z = \{V_1, W_3\}$, we have:

$$\begin{aligned} T_m &= \{W_2, V_2\}, \quad Z_m = \{V_1\}, \\ \{T_m \cup Z_m\}_m &= \{W_2, V_1, V_2\}_m = \{W_2, V_1\} \end{aligned}$$

We find that the tests of (9) and (10) translate into:

$$\{W_2, V_1\} \perp\!\!\!\perp Y | X, V_1$$

$$\{W_2, V_1\} \perp\!\!\!\perp Y | X, V_2, W_2$$

and are satisfied in the graph, thus implying $Z \approx T$.

That test would fail had we taken $Z = \{W_3, W_4\}$, which is inadmissible, because then we would have:

$$\begin{aligned} T_m &= \{W_2, V_2\}, \quad Z_m = \{W_4\}. \\ \{T_m \cup Z_m\}_m &= \{W_2, V_2, W_4\}_m = \{W_1, V_2\} \end{aligned}$$

and the requirement (9)

$$\{T_m \cup Z_m\}_m \perp\!\!\!\perp Y | X, Z_m$$

or

$$\{W_1, V_2\} \perp\!\!\!\perp Y | X, W_4$$

would not be satisfied.

Since conditions (9) and (10) are surrogate of the admissibility requirement in Theorem 2, we can pose them as graph-theoretic consequences of the back-door criterion:

Corollary 2. *In any DAG G , if two sets of nodes, T and S , satisfy the backdoor criterion, they must also satisfy the d -separation conditions expressed in (9) and (10).*

5 Empirical ramifications of c -equivalence tests

Having explicated the statistical implications of admissibility vis a vis c -equivalence, we may ask the inverse question: What can c -equivalence tests tell us about admissibility? It is well known that no statistical test can ever confirm or refute the admissibility of a given set Z (Pearl 2000, Chapter 6; 1998). The discussion of Section 4 shows however that the admissibility of two sets, T and Z , does have testable implications. In particular, if both sets fail the c -equivalence test, they cannot both be admissible. This might sound obvious, given that admissibility entails zero bias for each of T and Z (Eq. (5)). Still, Eq. (7) implies that it is enough for Z_m (or T_m) to fail the c -equivalence test vis a vis $\{Z_m \cup T_m\}$ for us to conclude that Z and T cannot both be admissible.

This finding can be useful when measurements need be chosen (for adjustment) with only partial knowledge of the causal graph underlying the problem. Assume that two candidate graphs recommend two different measurements for confounding control, one graph predicts the admissibility of T and Z , and the second does not. Failure of the c -equivalence test

$$T_m \approx \{T_m \cup Z_m\} \approx Z_m$$

can then be used to rule out the former.

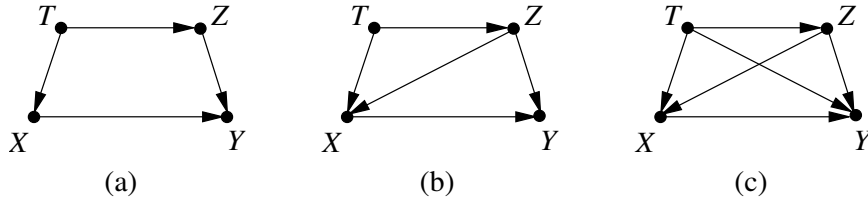


Figure 3: Failing the $T \approx \{T \cup Z\}$ test should reject Model (a) in favor of (b) or (c). Failing $Z \approx \{T \cup Z\}$ should reject Models (a) and (b) in favor of (c).

Figure 3 illustrates this possibility. Model 3(a) deems measurements T and Z as equally effective for bias removal, while models 3(b) and 3(c) deem T to be insufficient for adjustment. Submitting the data to the c -equivalence tests of Eq. (7) and Eq. (8) may reveal which of the three models should be ruled out. If both tests fail, we must rule out Models 3(a) and 3(b), while if only Eq. (8) fails, we can rule out only Model 2(a) (Eq. (7) may still be satisfied in Model 2(c) by incidental cancellation).

Of course, the same model exclusion can be deduced from conditional independence tests. For example, Models 3(a) and 3(b) both predict $T \perp\!\!\!\perp Y | X, Z$ which, if violated in the data, would leave Model 3(c) as our choice and behoove us to adjust for both T and Z . However, such tests are generally more expensive (and less reliable) than c -equivalence test, for they require conditioning on large sets of covariates. c -equivalence tests, on the other hand, can take advantage of propensity score methods (Rosenbaum and Rubin, 1983) which reduce the dimensionality of the conditioning set to a single scalar.

Figure 4 illustrates this potential more acutely. It is not easy to tell whether models (a) and (b) are observationally distinguishable, since they embody the same set of missing edges. Yet whereas Model 4(a) has no admissible set (among the observables), its contender, Model 4(b) has three (irreducible) such sets: $\{Z_1, W_1, W_2\}$, $\{W_1, W_2, Z_2\}$ are $\{V, W_1, W_2\}$. This difference in

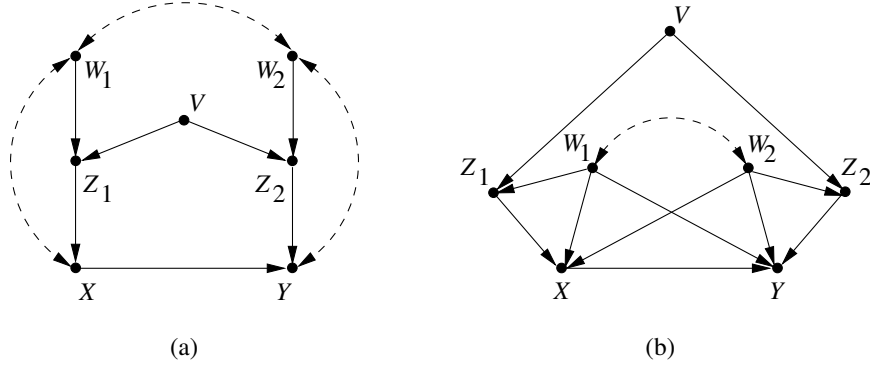


Figure 4: A model that is almost indistinguishable from that of Fig. 4(a), save for advertising one additional independency: $\{Z_1, V\} \perp\!\!\!\perp Y | X, W_1, W_2, Z_2$. It deems three sets to be admissible (hence c -equivalent): $\{V, W_1, W_2\}$, $\{Z_1, W_1, W_2\}$, and $\{W_1, W_2, Z_2\}$, and would be rejected therefore if any pair of them fails the c -equivalence test.

itself does not make the two models distinguishable (see Fig. 5); for example, $X \rightarrow Z \rightarrow Y$ is indistinguishable from $X \leftarrow Z \rightarrow Y$, yet Z is admissible in the latter, not in the former. However, noting that the three admissible subsets of 4(b) are not c -equivalent in 4(a) – their Markov boundaries differ – tells us immediately that the two models differ in their statistical implications. Indeed, Model 4(b) should be rejected if any pair of the three sets fails the c -equivalence test.

Visually, the statistical property that distinguishes between the two models is not easy to identify. If we list systematically all their conditional independence claims, we find that both models share the following:

$$\begin{array}{ll}
 V & \perp\!\!\!\perp \{W_1, W_2\} \\
 Z_1 & \perp\!\!\!\perp \{W_2, Z_2\} | \{V, W_2\} \\
 Z_2 & \perp\!\!\!\perp \{W_1, Z_1, X\} | \{V, W_2\}
 \end{array}
 \qquad
 \begin{array}{ll}
 X & \perp\!\!\!\perp \{V, Z_2\} | \{Z_1, W_2, W_1\} \\
 V & \perp\!\!\!\perp Y | \{X, Z_2, W_2, Z_1, W_1\}
 \end{array}$$

They disagree however on one additional (and obscured) independence relation, $Z_1 \perp\!\!\!\perp Y | X, W_1, W_2, Z_2, V$, that is embodied in Model 4(b) and not in 4(a). The pair (Z_1, Y) , though non-adjacent, has no separating set in the diagram of Fig. 4(a). While a search for such distinguishing independency can be tedious, c -equivalence comparisons tell us immediately where models differ and how their distinguishing characteristic can be put to a test.

For completeness, Fig. 5 present two models that are observationally indistinguishable, yet they differ in admissibility claims. Model 5(a) deems $\{T_1\}$ and $\{T_1, T_2\}$ to be admissible while Model 5(b) counters (a) and deems $\{Z_1\}$ and $\{Z_1, Z_2\}$ to be admissible. Indistinguishability requires that c -equivalence be preserved and, indeed, the relations $\{T_1\} \approx \{T_1, T_2\}$ and $\{Z_1\} \approx \{Z_1, Z_2\}$ are held in both (a) and (b).

This raises the interesting question of whether the discrimination power of c -equivalence equals that of conditional independence tests. We know from Theorem 5 that all c -equivalence conditions can be derived from conditional independence relations. The converse, however, is an open question.

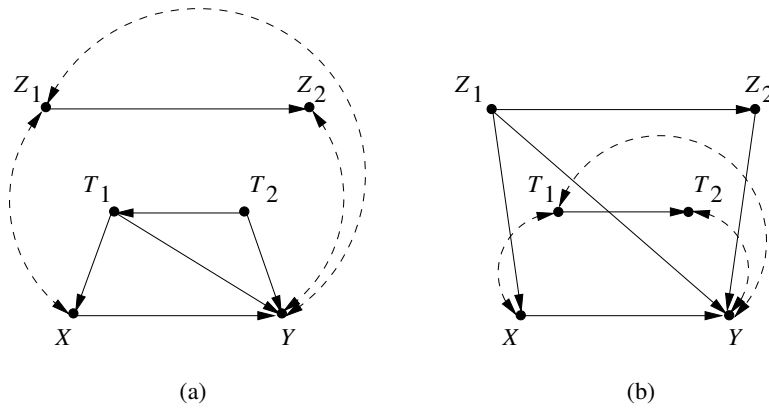


Figure 5: Two observationally indistinguishable models that differ in their admissible sets. Both confirm the c -equivalence $\{T_1\} \sim \{T_1, T_2\}$ and $\{Z_1\} \sim \{Z_1, Z_2\}$ but for different reasons.

6 Conclusions

Theorem 2 provides a simple graphical test for deciding whether one set of covariates has the same bias-reducing potential as another. The test requires either that both sets satisfy the back-door criterion or that their Markov boundaries be identical. Both conditions can be tested by fast, linear time algorithms, and could be used to guide researchers in deciding what measurement sets are worth performing. We have further shown that the conditions above are valid in causal as well as associational graphs, and translated them to conditional-independence claims. Finally, we postulate that c -equivalence tests could serve as valuable tools in a systematic search for graph structures that are compatible with the data.

A Appendix

We prove that, for any graph G , and any two subsets of nodes T and Z , we have:

$$(X \perp\!\!\!\perp Y | Z_m)_G \quad \& \quad (X \perp\!\!\!\perp Y | T_m)_G \Rightarrow (X \perp\!\!\!\perp Y | Z_m \cup T_m)_G$$

Where Z_m and T_m are minimal subsets of S and T , respectively that satisfy the left hand side of the expression above.

The following notation will be used in the proof: A TRAIL will be a sequence of nodes $v_1, \dots, v_k$ such that v_i is connected by an arc to $v(i+1)$. A collider Z is EMBEDDED in a trail if 2 of his parents belong to the trail it. A PATH is a trail that has no embedded collider. We will use the “moralized graph” test of Lauritzen et al. (1990) to test for d -separation (“ L -test,” for short).

Theorem 6. *Given a DAG and two vertices x and y in the DAG and a set of minimal separators between x and y $\{Z_1, \dots, Z_k\}$. The union of the separators in the set, denoted by $Z!$, is a separator.*

Proof.

We mention first two observations:

- (a) Given a minimal separator Z between x and y . If Z contains a collider w then there must be a path between x and y which is intercepted by w . This follows from the minimality of Z . If the condition does not hold then w is not required in Z .
- (b) It follows from (a) above that w as defined in a. and his ancestors must belong to the ancestral subgraph of x and y .

Let us apply the L -test to the triplet $(x, y | Z_1)$. As Z_1 is a separator, the L -test must show this. In the first stage of the L -test. The ancestral graph of the above triplet is constructed. By observation (b) it must include all the colliders that are included in any Z_i . In the next stage of the L -test. The parents of all colliders in the ancestral graph are moralized and the directions removed. The result will be an undirected graph including all the colliders in the separators Z_i and their moralized parents and their ancestors. In this resulting graph, Z_1 still separates between x and y . Therefore adding to Z_1 all the colliders in Z_i , $i = 1$ to k , will result in a larger separator. Adding the noncolliders from all the Z_i to Z_1 will still keep the separator property of the enlarged set of vertices (trivial). It follows that $Z!$ is a separator. End of proof.

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