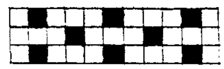
BRAIN TICKLERS

(Continued from page 51.)

5 Fill in the following cross-number puzzle with 13 different three-digit perfect squares. No leading zeros.



Read top-to-bottom and left-to-right as in a crossword puzzle.

—**Richard I. Hess,** *CA B '62*

B onus. If 12 one-ohm resistors are soldered together to form a cube, with each resistor being the edge of the cube, then the equivalent resistance of the network between two corners at the ends of a body diagonal is 5/6 ohm.

It is also possible to solder 32 one-ohm resistors together in the network equivalent of a four-dimensional hypercube. The vertices of a four-dimensional hypercube can be designated using the coordinates (w,x,y,z) where each variable has a value of 0 or 1. Then adjacent vertices are those in which only one variable has a different value; for instance, (0,1,0,1) and (0,1,1,1) are adjacent vertices. One can show that the equivalent resistance of the four-dimensional hypercube between a body diagonal, say (0,0,0,0) and (1,1,1,1), is 2/3 ohm.

Now, consider 192 resistors soldered together to form the network equivalent of a six-dimensional hypercube. What is the equivalent resistance between corners at the ends of a six-dimensional body diagonal?

—**John L. Bradshaw,** *PA A '82*

C omputer Bonus. Thirty-six is the smallest number, greater than one, that is both a triangular number and a perfect square. What are the next four numbers that are both triangular and a perfect square? Triangular numbers are numbers of the form n(n + 1)/2, the first few being 1, 3, 6, 10, and 15.

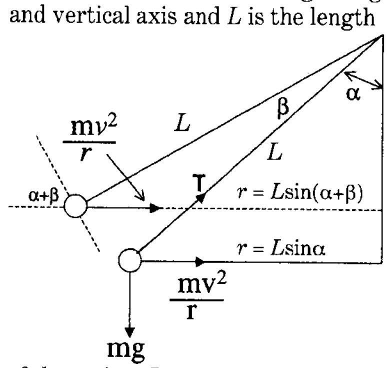
—*The Colossal Book of Mathematics* by Martin Gardner

Postal mail your answers to any or all of the Winter Brain Ticklers to Jim Froula, Tau Beta Pi, P.O. Box 2697, Knoxville, TN 37901-2697, or email to *BrainTicklers@tbp.org* only as plain text. The cutoff date for entries to the Winter column is the appearance of the Spring BENT in late March. The method of solution is not necessary.

We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Jim will forward your entries to the judges who are **H.G. McIlvried III,** *PA Γ '53;* **F. J. Tydeman,** *CA Δ '73;* **J.L. Bradshaw,** *PA A '82;* and the columnist for this issue, **D.A. Dechman,** *TX A '57.*

sum" for 4444, 4444^2, 4444^3, 4444^4, 4444^5, 4444^6, 4444^7, ... is 7, 4, 1, 7, 4, 1, 7, ... So, the 4444^{th} power will eventually truncate to 7. Alternatively, you can use congruence arithmetic: $4444 \equiv 7 \pmod 9$; $7^2 = 49 \equiv 4 \pmod 9$; and $7^3 = 343 \equiv 1 \pmod 9$. Therefore, $7^{4444} = 7^{3(1481)+1} = 7(7^3)^{1481} \equiv 7 \pmod 9$. From 4444log(4444) = 16210.7079, we see that $S(4444^{4444})$ will be 16,211 digits long with an expected average per digit of (0+1+2+3+4+5+6+7+8+9)/10 = 4.5 or around 72,950. Even the toughest case of all nines adds to 145,899 which has a second sum of 36 and a third sum of 9. So, it takes no more than three sums to reach a single digit.

Bonus. When the pendulum executing stable circular motion about a vertical axis (see figure) is perturbed with a tiny outward impulse, it executes small oscillations about the angle α with frequency $f = (1/2\pi)[(g/L)(3\cos^2\alpha + 1)/\cos\alpha]^{1/2}$, where α is the initial angle between the connecting string and vertical axis and L is the length



of the string. In our problem, L was given as one meter. Before the impulse, the horizontal component of the tension T in the string provides the centripetal force, mv_o^2/r_o, necessary to maintain circular motion, and the vertical component balances the force of gravity mg. Thus, $T\sin a = mv_o^2/r_o$ and $T\cos a = mg$. Eliminating T from these equations yields $v_o^2 = gr_o\sin\alpha/\cos\alpha = gL\sin^2\alpha/\cos\alpha$. As a result of the impulse, the angle a is increased to $a + \beta$, where β is small; v_o and r_o are also changed slightly to v and r, but the angular momentum remains approximately constant so that $mvr \approx mv_or_o$. After the impulse, the gravity component perpendicular to the string and acting to decrease β is $mg\sin(\alpha + \beta)$. The mass times centripetal acceleration component perpendicular to the string, also acting to decrease β, is $mv^2\cos(\alpha + \beta)/r$. Note that the horizontal component of tension becomes larger than mv_o^2/r_o for positive β and smaller than mv_o^2/r_o for negative β. This is the cause of the oscillatory motion. Substituting into the angular form of $F = ma$ yields $-mg\sin(\alpha + \beta) = m[Ld^2\beta/dt^2 - v^2\cos(\alpha + \beta)/r]$, where $Ld^2\beta/dt^2$ is acceleration in the plane of the figure and perpendicular to the string. Eliminating m, multiplying the last term by r^2/r^2, and rearranging gives $(v^2r^2/r^3)\cos(\alpha + \beta) - g\sin(\alpha + \beta) = Ld^2\beta/dt^2$. Now, $v^2r^2/r^3 = v_o^2r_o^2/r^3 = (gL\sin^2a/\cos a)(r_o^2/r^3) = g\sin^4a/[\cos a\sin^3(\alpha+ \beta)]$, upon substituting $r_o = L\sin a$, and $r = L\sin(a + \beta)$. Since β is small, $\sin(\alpha + \beta) \approx \sin\alpha + \beta\cos a$ and $\cos(\alpha + \beta) \approx \cos\alpha - \beta\sin\alpha$. Therefore, $g\sin^4a/[\cos a\sin^3(\alpha + \beta)] = g\sin^4a/[\cos a(\sin a + a\cos a)^3] = g\sin^4a/[\cos a(\sin^3a + 3a\sin^2a\cos a)] = g\sin^4a/[\cos a\sin^2a(\sin a + 3\beta\cos a)] = g\sin^4a(\sin a - 3\beta\cos a)/(\cos a\sin^4a) = g(\sin a - 3\beta\cos a)/\cos a = g\sin a(1 - 3a\cos a/\sin a)/\cos a$, where all terms involving higher powers of β have been discarded. Substituting these results into the angular $F = ma$ equation yields $g(\sin a/\cos a)(\cos a - \beta\sin a)(1 - 3\beta\cos a/\sin a) - g\sin a - g\beta\cos a = Ld^2\beta/dt^2$. Expanding, dropping higher powers of β, collecting terms, and dividing by L yields $-(g/L)(4\cos a + \sin^2a/\cos a)\beta = -(g/L)[(3g\cos^2a + 1)/\cos a]\beta = d^2\beta/dt^2$. This is the equation, $d^2\beta/dt^2 = -\omega^2\beta$, of a simple harmonic oscillator in the variable β, where ω is the angular frequency. The frequency f given above is obtained from $f = \omega/2\pi$.

Double Bonus. The function, $S(n) = 1(1!) + 2(2!) + 3(3!) + \ldots + n(n!)$ simplifies to $(n + 1)! - 1$. Let $T = 1! + 2! + 3! + \ldots + (n + 1)!$, which can be expressed as $1 + (1 + 1)1! + (1 + 2)2! + \ldots + (1 + n)n!$, which can also be expanded as $1 + 1! + 2! + \ldots + n! + 1(1!) + 2(2!) + \ldots + n(n!)$. So $T = 1 + T - (n + 1)! + S$. Therefore, $S = (n + 1)! - 1$. Also, you can observe that the values of S(n) for n = 1, 2, 3, 4, and 5 are 1, 5, 23, 119, and 719, from which you can deduce the general relationship.

NEW WINTER PROBLEMS

1 Al's job is testing bowling balls. He has two identical bowling balls and is to test their impact resistance by dropping them out of windows on various floors of a 100-story building. He is to determine from which exact floor a dropped bowling ball will shatter on impact with the pavement below. Al knows nothing about the strength of the balls. They may shatter when dropped from the first floor or not until dropped from the 100^{th} floor. What is the minimum number of ball drops needed to guarantee that Al can uniquely determine the floor from which the balls will shatter? Balls that do not shatter may be dropped again. Both balls may be destroyed during the test. Include a brief outline of how the testing is done.

—*How to Ace the Brain Teaser Interview* by John Kador

2 Our local Soggy Center Donut shop makes six different kinds of doughnuts—namely barbeque, garlic, pepperoni, jalapeno, broccoli, and onion. Each day I stop and buy a different selection of a dozen doughnuts. If I always buy at least one of each different kind, how many days will it take to exhaust all the possibilities? Assume that the shop always has at least seven of each different kind on hand.

—Adapted from *Introductory Combinatorics* by Richard A. Brualdi

3 If we write an integer in the decimal system, its representation either contains at least one digit 5 or it does not. Find the smallest and the largest values of N such that for the integers between 1 and N inclusive, exactly half contain at least one digit 5.

—Adapted from *Keys to Infinity* by Clifford A. Pickover

4 Given that TEN is one more than a perfect square that is divisible by 9, NINETY is divisible by 9, and there are SIX perfect squares between TEN and NINETY, what is the value of SENT? The usual rules for cryptics apply.

—Susan Denham in *New Scientist*

(Continued on page 53.)

Brain Ticklers

RESULTS FROM SUMMER 2009

Perfect

*Anderson, Paul M.	Son of member
*Bohdan, Timothy E.	IN Γ '85
*Celestino, James R.	NJ B '00
*Christianson, Kent B.	CA A '85
*Couillard, J. Gregory	IL A '89
*De Vincentis, Joseph W.	TX Γ '93
*Fenstermacher, T. Edward	MD B '80
*Ferguson, Adam B.	Non-member
*Fuemmeler, Jason A.	OH Θ '00
Mathews, Robert D.	AL E '94
*Meerscheidt, Kyle	Husband
*Rasbold, J. Charles	OH A '83

Other

*Alexander, Jay A.	IL Γ '86
Aron, Gert	IA B '58
Ballard, Jeffrey A.	FL E '05
*Berger, Toby	CT A '62
Bertrand, Richard M.	WI B '73
*Brule, John D.	MI B '49
*Christenson, Ryan C.	UT B '93
Collins, Paul A.	MA Z '86
*Doniger, Kenneth J.	CA A '77
Eksaa, Glenn	Non-member
Gluck, Frederick G.	CO B '67
Golembiewski, Steven L., Jr.	PA B '90
*Grabow, Benjamin P.	OK B '08
Harris, Kent	Non-member
*Herbert, Peter A.	OR Γ '10
Jenneman, Jeffrey H.	OK A '08
Jones, Donlan F.	CA Z '52
Kern, Peter L.	NY Λ '62
Lew, Thomas M.	TX Δ '84
*Mangis, J. Kevin	VA A '86
Mazeika, Daniel F.	PA B '55
McAuliffe, Lane	Non-member
*Norris, Thomas G.	OK A '56
Rentz, Peter E.	IN A '55
Schmidt, V. Hugo	WA B '51
Schorp, Katrina M.	TX Λ '08
*Schrum, Kevin D.	NE A '09
Schultz, Daniel	PR A '07
Sigillito, Vincent G.	MD B '58
*Smith, Charles J.	CT Γ '09
Spong, Robert N.	UT A '58
*Stribling, Jeffrey R.	CA A '92
*Strong, Michael D.	PA A '84
*Sutor, David	Son of member
*Thaller, David B.	MA B '93
Triantafillu, Dimos	AL B '07
Van Houten, Karen J.	ID A '76
*Venema, Todd M.	OH H '08
*Voellinger, Edward J.	Non-member
*Weinstein, Stephen A.	NY Γ '96
*Wendling, D. Greg	IL A '79
*York, Jeffrey A.	NC A '85

SUMMER REVIEW

Both problem #2 on the spinning sphere and problem #4 on the marbles and saucers had fewer correct answers than the Bonus problem. It appears that many people assumed that the great circle in problem #2 was picked before the shots were fired. The summer column proved to be very popular. We received entries from Tau Bates whose *alma maters* were in exactly half of the 50 states.

FALL SOLUTIONS

Readers' entries for the fall problems will be acknowledged in the Spring BENT. Meanwhile, here are the answers.

1 Al finished 8th, and his team scorers came in 2nd, 4th, 5th, and 6th. Careful analysis shows that there are only three possible outcomes for the winning and losing scores that yield only one possible scoring sequence each. Namely, 11 = 1+2+3+5 and 31 = 4+8+9+10 with 6,7,11,12 as non-scorers; 17 = 2+4+5+6 and 23 = 1+3+9+10 with 7,8,11,12 as non-scorers; and 13 = 1+3+4+5 and 29 = 2+8+9+10 with 6,7,11,12 as non-scorers. Note that the 8th place finisher is the only unique non-scorer, so the team's scores must have been 17 and 23, and Al was on the winning team.

2 It takes 27 ALCOHOLs to equal a HANGOVER. The only solution is 27(3451914) = 93201678. A simple computer program is helpful to save a lot of trial-and-error paperwork.

3 The sequence of inscribing an equilateral triangle in a unit circle, then inscribing a circle in the triangle, and repeating with a square then a circle, a regular pentagon then a circle, and so on, reaches a limit for the radius of the limiting circle of about 0.115. You can draw the inscribed regular polygon for the first few steps and readily observe that the radius of the next inscribed circle, r_{i+1}, equals to $r_i \cos\theta$ where $\theta = 180/n$ and n is the number of sides of the polygon. So, the answer is the limit of $\cos(60)\cdot\cos(45)\cdot\cos(36)\cdot\cos(30)\ldots\cos(180/n)$ as n approaches infinity. The answer quickly converges and can be found with a hand-held scientific calculator.

4 The expected distance of the closest point of three random points in a hemisphere to its base is $387R/2240$, where R is the radius of the hemisphere. This problem is similar to problems in our Spring 2006 and Fall 2007 columns, so you could have gone to *www.tbp.org* and looked at the solutions to those problems for guidance! You can do the math assuming a hemisphere of unit radius and then insert the radius back in at the end of the calculations, since the answer will be directly proportional to the radius.

Consider a thin slice through the hemisphere, parallel to, and a distance x from the base, and let $h = 1 - x$. The probability that a random point will fall within this slice is equal to the volume of the slice divided by the volume of the hemisphere, which, after simplification, equals $3h(2 - h)dh/2$. For the second point to be above the first point, it must be in the spherical segment bounded by the thin slice. The volume of this segment is $\pi h^2(3 - h)/3$ (see a math handbook), and the probability of a random point's being in this segment is its volume divided by the volume of the hemisphere which equals $h^2(3 - h)/2$. The probability of two random points being in the segment is $h^4(3 - h)^2/4$. Putting all this together, we get the expected value of h as $E(h) = (9/8)$ times the integral, from 0 to 1, of $h^6(2 - h)(3 - h)^2\, dh$ with an additional factor of 3 since any of the three random points can be the closest to the base. Integration give $E(h) = (9/8)(18/7 - 21/8 + 8/9 - 1/10) = 1853/2240$. $E(x) = 1 - E(h) = 1 - 1853/2240 = 387/2240$.

5 $S(S(S(4444^{4444}))) = 7$. The sum of the digits function is just our old friend, casting out nines, in disguise, which is based on the fact that the remainder when you divide a number by 9 is equal to the sum of the digits of the number, repeated if necessary to get a single digit number. You can quickly determine that the "triple