A Middle Way

Keywords Sums, Exponents, Patterns, Identities, Algebra

Abstract

Observations about patterns in number sequences. Associating them to topics in algebra. Identities. Sums of counting integers. Sums of those values squared. Expanding sum and difference of two terms squared. Using the mid-point in sequences. Expressions based on Pythagorean relationships.

1 Introduction

We often see sequences with a number pattern starting from a simple value - *e.g.,* $1, 2, 2^2, 3^2, 2^3$. This presents a different approach, moving away from the lowest. It instead looks at the middle of a finite sequence. This has the advantage of leading to algebraic expressions or identities, and may be valuable for class discussion.

A first finite sequence to view is from the *Pythagorean theorem:*

$$x^2 + y^2 = z^2 \tag{1}$$

The familiar Pythagorean triple satisfies equation (1):

$$3^2 + 4^2 = 5^2 \tag{2}$$

The trio $x = 3, y = 4, z = 5$ are *in sequence* or *consecutive numbers.* We call them *adjacent integers.* They are also *base values* since equation (2) shows squaring these integers.

Henceforth in any set of adjacent integers let's name the lowest number in the sequence its *beginning* and write it symbolically as b. Focussing on the beginning or least in value is one way to seek knowledge from number sequences.

2 Squares

Sometimes we may wish to avoid exponents as in the following representation of equation (2); here equation (3) is like its predecessor in that both show two square numbers adding up to a third. But the latter calls out values $9, 16, 25$ as special because they are squares in a sequence: there is no other whole square number in between, neither in (9, 16) nor (16, 25) intervals.

$$9 + 16 = 25 \tag{3}$$

Dealing with equation (2) base values we have $b = 3$, but just within the squared numbers in equation (3), $b = 9$.

Are squares a special case? That's Fermat's question about values of n in:

$$x^n + y^n = z^n \tag{4}$$

We turn this issue around here, asking instead about things that can be learned from simple powers, numbers in sequence, and reasoning about the beginning, middle and end of runs of values.

3 Cubes

Two books[1] present material relevant to moving from numbers to general statements. Polya [6] has two interesting numerical sequences that both begin with one ($b = 1$): the first shows "The sum of the

[1] See [6] and [2].

first n cubes is a square"; the second deals with the base values of those first squares, expressing them as "a remarkable regularity" involving sums of the first n integers:

$$
\begin{aligned}
1 = 1 &= 1^2 \\
1 + 8 = 9 &= 3^2 \\
1 + 8 + 27 = 36 &= 6^2 \\
1 + 8 + 27 + 64 = 100 &= 10^2 \\
1 + 8 + 27 + 64 + 125 = 225 &= 15^2
\end{aligned}
\tag{5}
$$

In the second Polya expands the base values:

$$
\begin{aligned}
1 &= 1 \\
3 &= 1 + 2 \\
6 &= 1 + 2 + 3 \\
10 &= 1 + 2 + 3 + 4 \\
15 &= 1 + 2 + 3 + 4 + 5
\end{aligned}
\tag{6}
$$

This leads to algebra for summing the first integers:

$$
\sum_{i=1}^{n} i = 1 + 2 + ... + n = n(n+1)/2 \tag{7}
$$

4 Resources

Other number sequences likewise lead to interesting expressions, while equation (2) is at the heart of Dunham's statement.[2]

[2]See [2] , p. 2.

> There is a tradition...that Egyptian architects used a clever device for making right angles. They would tie 12 equally long segments of rope into a loop ... Stretching five consecutive segments in a straight line ... and then pulling the rope taught ... they thus formed a rigid triangle with a right angle ... This configuration ... allowed the workers to construct a perfect right angle at the corner of a pyramid ...

We will need to use *the middle* both in sequences of numbers and in some shapes. To do that we need to examine some examples from varied sources.

The first source is a mathematical puzzle[3] stated this way:

> We are interested in finding a sequence of 2n + 1 consecutive positive integers, such that the sum of the squares of the first n + 1 integers equals the sum of the squares of the last n integers. The simplest such sequence is
>
> $$3^2 + 4^2 = 5^2$$

The first five instances[4] are:

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$

Table 1: Consecutive Squared-Integer Sums

[3]See [1].
[4]See [3]; also, [7] was consulted.

The second source goes beyond squares to cubes. In a textbook [4] the following statement[5] introduces an exercise: “The cube of every positive integer is a sum of consecutive odd integers.” Here are some instances:

$$\begin{aligned} 1 &= 1^3 \\ 3+5 &= 2^3 \\ 7+9+11 &= 3^3 \\ 13+15+17+19 &= 4^3 \\ 21+23+25+27+29 &= 5^3 \end{aligned}$$

Table 2: Cubes as Sums of Adjacent Odds

5 How to Proceed

One starts out wondering *where to begin* both in the puzzle case, and in decomposing a cube into adjacent odds. For instance if eleven integers are squared what is the least so that the lower six terms add to the upper five? Likewise if the cube of eleven is to be decomposed into the sum of adjacent odds, what is the first (lowest) one? In these cases we naturally are asking this question: *What is the b value?*

Let's take a more formal approach, applying algebra. Specifically, how would one write the result of any squaring of $2n+1$ adjacent bases and then summing the results? We can see our five Table 1 cases as instances of this formal expression in b and n:

$$b^2 + (b+1)^2 + ... + (b+n)^2 = (b+n+1)^2 + ... + (b+2n)^2 \qquad (8)$$

[5] See [5].

Equation (8) displays counting on each side of the equal sign[6] Simpler and similar relations result from using m for *middle value, i.e.,* either the *base* or a number just left of the equal sign. To see that start from these three examples:

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

Their general form is:

$$(m - n) + ... + m = (m + 1) + ... + (m + n) \tag{9}$$

Equation (9) supports restating equation (8) in terms of m:

$$(m - n)^2 + (m - n + 1)^2 + ... + m^2 = (m + 1)^2 + ... + (m + n)^2 \tag{10}$$

6 Obtain a Rule

There is one more term on the left than the right in both equations (9) and (10). Except for the middle value [m, or m^2] every RHS term has a similar one on the LHS. This lets us solve for m in each case. For equation (9), collect m-terms to find:

$$m = (n + 1)m - n(m) = 2\sum_{i=1}^{n} i = n(n + 1) \tag{11}$$

The equation (10) squares simplify through n LHS/RHS term applications of this useful algebraic fact:

$$(g - h)^2 + 4gh = (g + h)^2 \tag{12}$$

[6] Below, for left-hand (right-hand) side we write LHS (RHS).

Separating the cross-products from m^2 gives:

$$m^2 = 4\sum_{i=1}^{n} mi \tag{13}$$

So:

$$m = 4\sum_{i=1}^{n} i = 2n(n+1) \tag{14}$$

We've shown simple multiples of summing the first n integers yield two sought middle values m. Twice that sum suffices for equation (7), the linear case. Four times it is needed in the case of the quadratic expressions that extend equation (2) to more terms.

In both cases, once one has the middle value m it is an easy matter to get a corresponding beginning value b.

7 Inside a Solid

The *middle* offers a way to understand cubes as sums of adjacent odd numbers. At first obtaining the sequence of adjacent odd numbers that sum to some base c cubed, *e.g.,* $c^3 = 17^3$ seems daunting. Looking at the cube via a process based on the middle m yields a solution.

If finding adjacent odds that sum to a cube is obscure, look at the base c. We know there are two kinds of natural numbers: odds and evens. As in the Pythagorean-extension, this works: *go to the middle of the cube.* Consider the two instances where the cube is either of 2 or 3. Whenever c is even, the mid-point m is between levels of a physical cube. In the case 2^3, think of the eight octants in a three-space Cartesian coordinate system. If dealing with 3^3, use a 3-dimensional tic-tac-toe model. Either way m tells which odd integers begin the sequence that sums to c^3. One simply proceeds from the middle outwards.

Let's look at two instances: $2 \times 2 \times 2$ and $3 \times 3 \times 3$ cubes. With even base c, subtract and add one to the square values on either side of the middle. Then continue the process, subtracting/adding from the prior reduced value, three, five, ..., until reaching zero. (The separation into odds and evens, and this suggested process, could lead to introducing mathematical induction.)

The same sort of thing takes place with c odd except that now the middle is itself an odd number c^2. Hence the first two values are $c^2 - 2, c^2 + 2$ as seen in the $c = 3$ second instance here:

$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3$$

8 Square Sum of Summed Squares

The apex of a square-based pyramid is at the middle in many ways, and whether at modern and ancient sites[7] or imaged on currency, they are part of the general public's consciousness. With an 1875 conjecture about pyramids, a surprising uniqueness proof completed less than a century ago [8], connects them to the above middle-way instances and algebraic expressions.

We can replace the physical examples by a model created by stacking spheres: oranges, marbles, golf balls or cannon balls. Pile them up to make a solid with one at the top, a four in a square immediately below it, nine likewise arranged next down, and so forth to the n-th level (n^2). Watson resolved the question of choosing n so that for some integer m the next equation holds:

$$1^2 + 2^2 + ... + n^2 = m^2 \tag{15}$$

[7] Paris (Le Louvre), Las Vegas (Luxor Hotel), and Giza.

For these consecutive squares summed beginning at $b = 1$ Watson showed that the only square total, 4900, happens at a final base value of 24,[8] by applying:

$$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6 \quad (16)$$

9 Conclusion

Observations about number patterns and relating them to mid-sequence values could be starting points for narrations about general algebraic relationships like equations (16), (7) and (11). Interconnections between Pythagorean triples, pyramids, and sequences derived from questions about squares and cubes could be used to start student inquiries using calculators and web-based materials.

10 Acknowledgement

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11 Captions

Table 1. Consecutive Squared-Integer Sum Relationships
Table 2. Cubes as Sums of Adjacent Odds

References

[1] Brain ticklers. The Bent of Tau Beta Pi, Spring 2004, p. 44; available on the world-wide web as http://www.tbp.org/pages/publications/BTs/Sp04.pdf, 2004.

[8] See [8] or (web page) [9].

[2] William Dunham. *Journey Through Genius: The Great Theorems of Mathematics.* John Wiley & Sons, Inc., 1990.

[3] Martin Gardner. *The Numerology of Dr. Matrix.* Simon and Schuster, NY, 1967.

[4] Thomas L. Naps. *Introduction to Data Structures and Algorithm Analysis, 2nd Edition, p. 50, prob. 4.* West Publishing Co., St. Paul, Minnesota, 1992.

[5] Nicomachus. Introduction arithmetica, 50.

[6] G. Polya. *How to Solve It.* Doubleday Anchor Books, 1957.

[7] T. H. Beldon. Runs of Squares. *The Mathematical Gazette*, pages 334–335, December 1961.

[8] Watson, G. N. The Problem of the Square Pyramid. *Messenger. Math. 48*, pages 1–22, 1918.

[9] Weisstein, E. W. Square Pyramidal Number. MathWorld; http://mathworld.wolfram.com/SquarePyramidalNumber.html, 1999.