Patterns in Numbers

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Abstract

Identities involving sums, and *sums of squares.* Relation of pattern thinking to algebra concepts. Consecutive-integer relationships. Mathematical exposition using patterns and symmetry.

1 Introduction

The mathematics curriculum mentions *Pythagorean theorem* and introduces numeric instances, particularly that based on 3, 4, and 5. Many would not imagine that there are other relationships involving sums of squared consecutive integers. Yet a mathematical puzzle [magazine column [1]] contained this statement:

> We are interested in finding a sequence of 2n + 1 consecutive positive integers, such that the sum of the squares of the first n + 1 integers equals the sum of the squares of the last n integers. The simplest such sequence is
>
> $$3^2 + 4^2 = 5^2 \tag{1}$$

That led me to find and review or consider [2], an unavailable letter [3] mentioned there, and [4]. The last presents the first four such equations involving summing sequences of adjacent integers-squared.

Using an observation by [3], [2] presents the first five such equations.

Applying pattern thinking leads to more general understanding. Ultimately, that approach goes through straight-forward algebra to explain the practical observations that gave rise to such puzzles.

There material proceeds from two basic pattern cases. The first involves extending (1) to two more sum of squared consecutive integer cases – see equations (2) and (3) near the beginning of the next section. The second case is somewhat simpler because only adjacent integers are present, not their squares. Examples appear in equations (4)-(6) below.

2 Observations

What value begins the series of adjacent squares? Gardner wrote that Linton [3] gave a formula for the lowest value in an equation like the two that follow:

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2 \tag{2}$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 \tag{3}$$

Instead of the lowest value, view this through pattern thinking. First look at the three equations (1)-(3) in terms of symmetry. To go there, note that in each case the number of entries on the left side is always one more than how many are right of the equals sign. [Here and throughout we adhere to a convention that the left hand side of such equations is always the expression with the most entries.] There is symmetry about the largest of the numbers (before squaring) on the left hand side. In essence the symmetry comes from counting down from that number on the left, and up from it on the right.

Suppose one divides the numbers that are squared into three groups: those lower than the value closest to, and left of, the equals sign; that number itself; and those on the right side of the equality.

In (1)-(3), but actually any consecutive squares equation, all terms are derived by counting from the particular value we just isolated (one of three groups). That value is always immediately left of the equals sign when there is one more term at its left than on the right. If we call the isolated value (with location just left of the equals sign when the larger number of consecutive squares is on the left side) *the pivot,* numbers to be squared on the left are obtained by counting down from it (those at right, up).

There are other expressions not involving squares that are very similar to the above equations, for example (4)-(6). In each case (1)-(6) there is a single number from which all others in that equation are found by counting down or counting up.

Equation (7) states algebraically what takes place in the equations (4)-(6), using a to represent the largest offset from the pivot term t.

$$1 + 2 = 3 \tag{4}$$

$$4 + 5 + 6 = 7 + 8 \tag{5}$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15 \tag{6}$$

$$(t - a) + (t - a + 1) + ... + t = (t + 1) + ... + (t + a) \tag{7}$$

Every term on the RHS of (7) has (after the letter t) some value between one and a; hence there are a-many entries there. Similar inspection of the terms on the LHS of (7) shows that there are *a + 1* terms left of the equals sign.

Expanding the elements in (7) lets us collect all terms with t's on the LHS, and all without t's on the RHS. That leads to these two equalities [equation (8)].

$$(a+1)t - a(t) = t = 2(sum - of - numbers - from - 1 - to - a) \tag{8}$$

Rewriting *sum-of-numbers-from-1-to-a* using summation notation, and division through by 2 yields this restatement of equation (8):

$$t/2 = \sum_{i=1}^{a} i = a(a+1)/2 \tag{9}$$

To check, substitute $a = 3$ in equation (9) and find the equation (6) pivot value.

Equation (9) is often demonstrated by writing two rows of numbers 1 through a, one below the other but in reverse order. Vertically adding similarly-located values gets just a entries that all are *(a + 1)* in value, hence [since that addition yields *sum-of-the-numbers-from-1-to-a* twice] :

$$2\sum_{i=1}^{a} i = a(a+1) \tag{10}$$

We now know that in an expression like (4)-(6) a number just left of the equals sign (side with larger number of terms) has value t ; that t begins a counting down and up process that is behind such an equality; and that t an be calculated from the number of terms on the right (fewer entries) side, a , by:

$$t = a(a+1) \tag{11}$$

Suppose we call the lowest value in an equation of the same form as (1)-(3) b. Then Linton's formula becomes:

$$b = a(2a+1) \tag{12}$$

3 Squares

Stimulated by [1] this section examines the general nature of equations in consecutive squares like (1)-(3). Instead of seeking the lowest value b in such expressions consider just *finding the pivot value.* The reasoning begins with an expression analogous to (7) stating the consecutive square property:

$$(t-a)^2+(t-a+1)^2+...+t^2=(t+1)^2+...+(t+a)^2 \tag{13}$$

There again are three groups of numbers being squared. One group is just *the pivot,* t . The other two, are numbers below it, and those that are higher. The values going to be squared in these last two groups come from differences or sums to t (they count down or up from it). Because there are similar terms from each group, differing only by changing a minus to a plus, reasoning employs the following:

$$(c-d)^2=c^2-2cd+d^2=(c^2+2cd+d^2)-4cd=(c+d)^2-4cd \tag{14}$$

The property displayed in (14) applies to (13) and creates a simple result. When two numbers are added or subtracted and the result is squared, they yield the same squared terms and only differ in the sign of their cross-products. Every *minus* element in (13) has a corresponding *plus* term except t^2 itself. This means that (13) can be rewritten:

$$t^2=4\sum_{i=1}^{a} ti \tag{15}$$

This simplifies to:

$$t=4\sum_{i=1}^{a} i \tag{16}$$

The result in (16) can be restated by means of the expression for sums of integers from *1* to *n,* (10). Applying (10) to (16) gives twice the linear case result [equation (11)]:

$$t=4\sum_{i=1}^{a} i=2a(a+1) \tag{17}$$

4 Values

This section develops relationships among numbers that could have led to the general conclusions about sequences of adjacent numbers squared and summed. There are two basic or elementary notions, the idea of Linton about the lowest number starting the sequence, and that presented here involving the number of entries counting up from the pivot.

Since observation and awareness of pattern proceeds from numeric results, and using words for concepts precedes algebra's symbols, we introduce nomenclature for these notions.

Instead of a let *order* signify *smallest number of terms* in a relationship between two sets of summed squares on consecutive values. The symbol b can stand for *base* or *bottom,* each consecutive square equation's *lowest-value-being-squared.*

The observed values for order and base of equations (1)-(3) follow in tabular form (Table 1). Notice that the product of *order* and *number of entries* is the *base* (*entries* refers to the total number of terms or elements in an equation). Stated algebraically, this is Linton's formula, (12), namely $a(2a + 1) = b$.

order	*base*	*number of entries*
1	3	3
2	10	5
3	21	7

Table 1: Product Pattern

Another table, Table 2, shows symmetry about the pivot. It also displays pattern relationships between characteristics of different order equations.

(count	down	from)	*pivot*	=	(count	up	from)
		3	*4*	=	5		
	10	11	*12*	=	13	14	
21	22	23	*24*	=	25	26	27

Table 2: Symmetry Pattern

Notice the result from multiplying two items and dividing by a third. Those to be multiplied are *number of left side terms* and *largest value on right.* Division is by *number of right side terms.* The result is exactly *base of next-higher-order equation.*

Also, notice that right side counting up from a pivot makes the largest value be $t + a$. In algebraic terms:

$$(t_a + a_a)(a_a + 1)/a_a = b_{a+1} \tag{18}$$

Since (12) holds in general, (18) can be rewritten using $b_{a+1} = (a+1)(2a+3)$:

$$(t + a)(a + 1)/a = (a + 1)(2a + 3) \tag{19}$$

Algebraic manipulation here again leads to $t = 2a(a + 1)$ as in (17).

5 Square Sum of Summed Squares

A series of consecutive squares summed begins with unity. As the number of elements summed increases, an interesting fact appears. In just one case will the sum be square. This occurs only when the last number squared is twenty-four [5] . (For more detail on this question see [6] ; proof is beyond the scope of this item.) Another way to view this is that spherical objects are arranged in a pyramid with square cross-section at every level. Then the question Watson resolved is that twenty-four is the sole number of levels for the pyramid with apex one, such that all the elements completely fill a flat square.

Watson's proof depends on another summing squares relationship. It equates the sum of squares beginning at one and progressing through the integers until a value *n,* to the product of three factors in n:

$$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6 \tag{20}$$

Although 12, 18, 24, 30 and 36 all are evenly divided by 6, only one of these can be an n so that (20) is square: $n = 24$. The following table shows that the others do not result in a square, while 24 does.

n	*n + 1*	*2n + 1*	*RHS eq. (20)*
12	13	25	650
18	19	37	2109
24	25	49	4900
30	31	61	9455
36	37	73	16206

Table 3: Product for Squares Summed

The way to establish (20) begins with the first few values of such a sum as n increases from unity. Direct computation of the (20) right-side product, gets 1, 5, 14, 30, 55 respectively from $n = 1, 2, 3, 4, 5$. Again a little work verifies things like $1^2 = 1, 1^2 + 2^2 = 5$, showing that (20) holds for some values, hence it can be proven by a simple process.

The process involves this observation: if it holds at say *n implies or causes* it to also be valid at *(n+1)*, then as long as it is true for some case n it must be true for any *n.* Its truth at the five numerical values indicates a pattern that makes it worth investigating the relationship:

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \tag{21}$$

Evaluating the two expressions in (21) by multiplying out terms gives the same result. Hence (20) is true in general.

One can add the squared terms for integers from one to $n = 24$ and verify that the same result occurs as in (20), namely 4900. Of course that is a square, specifically, $4900 = 70^2$. Watson demonstrated that there is just $n = 24$, as a value n so a sum of numbers squared from 1 to n itself is square.

6 Conclusion

Mathematics comes from observations about patterns in numbers and nature. This paper describes such relationships and presents two identities with the goals of sustaining classroom interest and stimulating continued mathematical learning.

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References

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