Cramer-Rao Bound for Frequency Estimation in Coherent Pulse Train with Unknown Pulses

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Abstract—Previous results for Cramer-Rao lower bound (CRLB) of the frequency of a coherent pulse-train make severely simplifying assumptions not always valid in real-world applications. Here the CRLB is derived for the case that the pulse is arbitrary but unknown. This paper poses and answers the important question: does a priori knowledge of the pulse shape confer a fundamental advantage (i.e., lower CRLB) as compared to the case of known pulse shape? The surprising answer is that such prior knowledge provides very little advantage in any expected real-world scenarios.

Index Terms—Cramer-Rao Lower Bound, Frequency Estimation, Doppler Shift, Coherent Pulse Train, Emitter Location, Geolocation, Radar Processing.

I. INTRODUCTION

Passive location of a stationary emitter with unknown frequency from a single moving platform is an important problem that has been investigated recently [1]–[11]. The importance of single-platform methods stems from the desire to provide accurate location of emitters from a single aircraft for tactical and/or strategic uses: single aircraft ability provides flexibility over multi-platform methods [6] and therefore there has been much practical interest in the development and testing of single-platform methods [2], [3]. All the single-platform approaches that have been proposed consist of measuring, at instants within some interval of time, one or more signal features that depend on the emitter’s location (e.g., Doppler shift, phase between two antennas, etc.), and then processing them to estimate the emitter location. To evaluate the location accuracy it is first necessary to evaluate the accuracy of the feature estimation; here the focus is on deriving the Cramer-Rao lower bound (CRLB) for the estimation of the Doppler-shifted frequency of a coherent pulse-train signal. Prior results for this CRLB have focused only on the case where either the simple case of a pure sinusoid that is turned on and off [12] or for the case where the pulses are assumed to be known [13].

This paper poses and answers the important question: does a priori knowledge of the pulse shape confer a fundamental advantage (i.e., lower CRLB) as compared to the case of known pulse shape? It is well known that, in general, failure to account for the unknown nature of the underlying signal can lead to an overly optimistic CRLB; an excellent example of this has been given by Yeredor and Angel [14]. Similarly, since it is also well-known that for classic sinusoidal frequency estimation, failure to consider an unknown phase will give an overly optimistic CRLB on the frequency estimation (see Example 3.14 and Problem 3.11 in [15]) it is natural to expect in the problem of interest here that assuming that the pulse’s modulation is known would impact the CRLB on the frequency estimate. However, the surprising answer for the frequency estimation of a coherent pulse train is that such prior knowledge provides very little advantage.

Section II establishes the appropriate signal model and then derives from it the CRLB on the frequency of a coherent pulse train. As in [13], the result is expressed in terms of some meaningful characteristics of the underlying pulse and the overall pulse train. Section III provides a detailed discussion and interpretation of the results including the presentation of numerical results for several typical pulse trains. The derived results are compared to those in [13] and – quite unexpectedly – the results are found to be numerically nearly the same, except for some cases unlikely to occur in practice; there is an extra term in the denominator of the result in [13] but is is negligible in most cases of interest. Without these results it would be impossible to tell (and very unexpected) that the two cases would have virtually identical bounds.

II. CRLB FOR ESTIMATION OF DOPPLER IN COHERENT PULSE TRAIN

Many modern radars emit so-called coherent pulse trains where there is a single underlying sinusoid that is turned on and off by a multiplicative baseband pulse train, where the underlying pulse can be nonrectangular and include phase/frequency modulation [11], [16]. It is the coherency that can be exploited to yield frequency estimates accurate enough for use in locating the emitter [11]. A complex-valued coherent pulse train consisting of K pulses received from an emitter can be modeled by
\[ s(t) = e^{j2\pi ft} \sum_{k=0}^{K-1} p(t-T_k) \]  

where \( p(t) \) is a single pulse (possibly complex), \( f \) is the Doppler-shifted frequency in hertz, and the \( T_k \) are called the pulse times. Any phase on the carrier can be included here in \( p(t) \). Without loss of generality we can model the complex \( p(t) \) as

\[ p(t) = A(t)e^{j\theta(t)}, \]

where \( A(t) \) is a real-valued envelope and \( \theta(t) \) is the pulse phase modulation. In practice, the pulse \( p(t) \) is generated as a baseband pulse and therefore it is generally valid to assume that \( f_p = 0 \) and we assume that here; this is consistent with Class III in [16] and describes modern pulse Doppler radars. Indeed, if there were such a constant frequency term in \( \theta(t) \) it should be grouped with the constant frequency term in the carrier. These conditions ensure the decomposition in (1) is unique and yields a well-posed frequency estimation problem for the parameter \( f \) in (1).

To derive the CRLB for the Doppler shift on a received coherent pulse train, we assume the transmitted signal is an unknown deterministic signal and thus needs to be considered to be estimated when computing the CRLB. This view is similar to that done for a different scenario in [14]. We model the received coherent pulse train as in (1) where \( p(t) \) is the unknown single pulse modeled as in (2), \( f \) is the Doppler-shifted frequency, and \( T_k \) are the pulse times; the received signal is assumed to be corrupted by additive white Gaussian noise. We assume that the received signal is sampled with sampling intervals \( T_s \) to yield its discrete-time versions and assume that the sampling rate is high enough that we can approximately define \( l_k = T_k / T_s \) as an integer\(^1\). Then, we have

\[ r[n] = s[n] + w[n] \]

\[ = e^{j2\pi f_{ns}} \sum_{k=0}^{K-1} p[n-l_k] + w[n], \quad n = 0,1,...,N-1 \]  

where \( K \) is the number of pulses, \( N \) is the number of samples, \( \omega \) is the Doppler-shifted frequency in \( \text{rad/sample} \), \( p[n] \triangleq p(nT_s) \) with \( p(t) \) as defined in (2), and \( w[n] \) is complex zero-mean white Gaussian noise with variance \( \sigma^2 \). Now, we can rewrite (3) in vector form as

\[ r = s + w \]  

where the vector \( s = [s[0] s[1]...s[N-1]]^T \) is the noise-free signal vector at the receiver, \( r = [r[0] r[1]...r[N-1]]^T \) is the observation vector and \( w \) is complex-valued white Gaussian noise vector. Since \( s \) is a deterministic signal, it sets the mean of the received vector and has no impact on the variance. Thus, \( r \) is a complex Gaussian vector with mean \( \mu = s \) and covariance \( C = \sigma^2 I_N \) where \( I_N \) is the \( N \times N \) identity matrix.

Assuming that \( p = [p[0] p[1]...p[M-1]]^T \) is the \( M \times 1 \) vector of the main pulse samples, we can define two vectors \( p_r \triangleq \text{Re}\{p^T\} \) and \( p_i \triangleq \text{Im}\{p^T\} \) as the real part and the imaginary parts of the pulse vector. We can also define the \((K-1)\times1\) vector \( l \triangleq [l_1, ... , l_{K-1}] \) as the pulse time vector, which are unknown since the precise pulse spacing is not known to the receiver (similar to the case of delay estimation in Example 3.13 in [15]). Although we are only interested in the CRLB of the parameter \( \omega \), there are other unknown deterministic parameters, namely \( l_1, l_2, ... , l_{K-1} \) (or vector \( l \)) and the complex pulse vector \( p \), that must be considered as nuisance parameters; we assume that \( l_0 = 0 \) since otherwise the CRLB computations become ill-posed – alternatively, this means that what matters is the spacings between pulses and not the absolute placement of the pulse train as a whole. We define the parameter vector \( \theta \) with \((2M+K)\) elements containing all real-valued unknown parameters\(^2\) as

\[ \theta = [p_r, p_i, l, \omega]^T \]  

\[ \text{Fisher information matrix (FIM) for real-valued parameters estimated from a complex vector that is signal plus white Gaussian noise with variance} \quad \sigma^2 \text{is}\]  

\[ \text{FIM} = \frac{2}{\sigma^2} \text{Re}\left\{ \frac{\partial s}{\partial \theta} \right\}^H \left\{ \frac{\partial s}{\partial \theta} \right\} \]

In order to simplify the expressions, we define some matrices that will be used frequently:

\[ W \triangleq \text{Diag}\{e^{j\theta_0}, e^{j\theta_1},...,e^{j\theta(N-1)}\} \]

\(^1\) It is assumed that the noise remains white at this level of sampling rate. Such an assumption is commonly made in similar developments; e.g., see Example 3.13 in [15].

\(^2\) Note that the parameters in vector \( l \) are integers and might be expected to lead to trouble when taking derivatives for the CRLB. It is possible to handle these alternatively as real-valued time shifts as was done in [13]. We have chosen to leave them explicitly as integers but treat them as real-valued when taking derivatives – the result is consistent with the results in [13].
\[ A \triangleq \begin{bmatrix} I_{M \times M} & 0_{M \times (l_k - M)} & I_{M \times M} & 0_{M \times (l_k - M)} & \cdots \\ 0_{M \times (l_k - l_{k-1} - M)} & I_{M \times M} & I_{M \times M} & 0_{M \times M} & \cdots \end{bmatrix} \]
\[ 0_{M \times (l_k - l_{k-1} - M)} \]

where \( L_k \triangleq \sum_{k=1}^{K-1} l_k \) and the last equality comes from the fact that \( A^T N A = L_k I + KM \).

\[ \tilde{P} \triangleq \begin{bmatrix} p'(\Delta n - T_1) & p'(\Delta n - T_2) & \ldots & p'(\Delta n - T_{K-1}) \end{bmatrix} \]

where \( p'(\Delta n - T_k) \triangleq \frac{dp(t)}{dt} \mid_{t=\Delta n - T_k} \)

\[ N \triangleq \text{Diag}\{0, 1, \ldots, N - 1\}; \]
\[ M \triangleq \text{Diag}\{0, 1, \ldots, M - 1\} \]

According to (7) and (8), we can easily show that

\[ s = WA p \quad \text{(11)} \]

where \( s \) is the signal vector and \( p \) is the \( M \times 1 \) vector of the pulse samples as defined before. Now using the equations (7)-(11), we can write

\[ \frac{\partial s}{\partial \theta} \left[ \frac{\partial s}{\partial p_r} \right]_{N \times (2M + K)} = \left[ WA \quad jWA \quad -W\tilde{P} \quad jNs \right] \quad \text{(12)} \]

According to (6) and (12), the elements of the Fisher Information matrix are given by

\[ \left( \frac{\partial s}{\partial p_r} \right)^H \left( \frac{\partial s}{\partial p_r} \right) = (WA)^H (WA) = A^H A = KI_{M \times M} \]
\[ \left( \frac{\partial s}{\partial p_i} \right)^H \left( \frac{\partial s}{\partial p_i} \right) = (WA)^H (jWA) = jA^H A = jKI_{M \times M} \]
\[ \left( \frac{\partial s}{\partial \omega} \right)^H \left( \frac{\partial s}{\partial \omega} \right) = (WA)^H (-W\tilde{P}) = -A^T \tilde{P} \]
\[ \left( \frac{\partial s}{\partial \theta} \right)^H \left( \frac{\partial s}{\partial \theta} \right) = (WA)^H (jNs) = jA^T W^H N s = (jA^T W^H N)(WAp) = j(A^T N A) p = j(L, I + KM) p \]

Consequently, the Fisher Information Matrix will be,

\[ FIM = \begin{bmatrix} \frac{2}{\sigma^2} \text{Re} \left( \frac{\partial s}{\partial \theta} \right)^H \left( \frac{\partial s}{\partial \theta} \right) \end{bmatrix} = \]

\[ \begin{bmatrix} \text{Diag} N & \text{Diag} M \\ -\text{Diag} M & \text{Diag} N \end{bmatrix} \]

\[ \begin{bmatrix} KI_{M \times M} & 0_{M \times M} & -A^T \tilde{P} & jA^T N A \\ -jKI_{M \times M} & KI_{M \times M} & jA^T \tilde{P} & A^T N A \\ -p^H A & -p^H \tilde{P} & E_I(K-1) & -j\tilde{P}^H N A p \\ -j p^H A^T N A & p^H A^T N A & j p^H A^T N \tilde{P} & s^H N^2 s \end{bmatrix} \]
Now, we can write the matrix $FIM$ as a block matrix as

$$FIM \triangleq \frac{2}{\sigma^2} \begin{bmatrix} G & B \\ C & D \end{bmatrix}$$

$$G \triangleq \begin{bmatrix} KI_{M \times M} & 0_{M \times M} \\ 0_{M \times M} & KI_{M \times M} \end{bmatrix};$$

$$B \triangleq \begin{bmatrix} -A^T \bar{P}_i & -A^T N A p_r \\ -p_i A^T N A & p_i A^T N A \end{bmatrix};$$

$$C \triangleq \begin{bmatrix} -\tilde{P}_i A & -\tilde{P}_i A \\ -p_i A^T N A & p_i A^T N A \end{bmatrix};$$

$$D \triangleq \begin{bmatrix} \tilde{E}I_{(K-1)} & \text{Im}(\tilde{P}^H N A p) \\ -\text{Im}(p^H A^T N \tilde{P}) & s^H N^2 s \end{bmatrix}. $$

Using the Block Matrix Inversion formula [17] and taking advantage of diagonality of upper-left block matrix in (20), yields the inverse of the $FIM$ as

$$FIM^{-1} = \frac{\sigma^2}{2} \begin{bmatrix} G^\dagger + G^\dagger B(D - CG B)^\dagger CG^{-1} & -G^\dagger B(D - CG B)^\dagger \\ -(D - CG B)^\dagger CG^{-1} & (D - CG B)^\dagger \end{bmatrix}$$

Thus, the CRLB matrix of $l$ and $\omega$ is given by,

$$\text{CRLB}_{l,\omega} = \frac{\sigma^2}{2} \times (D - CG B)^{-1} \tag{21}$$

As shown in the appendix, the FIM of $l$ and $\omega$ is simplified to

$$FIM_{l,\omega} = \text{CRLB}_{l,\omega}^{-1} = \frac{2}{\sigma^2} \times (D - CG B)$$

$$= \frac{2}{\sigma^2} \times \begin{bmatrix} \tilde{E}(I_{(K-1)} - (1/K)I_{(K-1)\times(K-1)}) & \text{Im}(p^H p) \tilde{L} \\ \text{Im}(p^H p) \tilde{L}^T & (L_2 - L_1^2 / K)E_{0 p}I_{K\times K} \end{bmatrix} \tag{22}$$

where $I_{(K-1)\times(K-1)}$ is a $(K-1)\times(K-1)$ matrix with 1 as all elements, $E_{0 p} \triangleq p^H p = \sum_{n=0}^{M-1} |p(n\Delta)|^2$, $\tilde{E} \triangleq \sum_{n=0}^{M-1} |p'(n\Delta)|^2$, $L_1 \triangleq \sum_{k=1}^{K-1} l_k$, $L_2 \triangleq \sum_{k=1}^{K-1} l_k^2$ and

$$\tilde{L} \triangleq [l_1 - L_1 / K \quad l_2 - L_1 / K \quad \ldots \quad l_{K-1} - L_1 / K]^T.$$

Thus, we can again use the block matrix inversion formula to find the CRLB matrix of $l$ and $\omega$. Assume that

$$FIM_{l,\omega} = \frac{2}{\sigma^2} \times \begin{bmatrix} \tilde{G} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$$

Then, we have

$$\text{CRLB}_{l,\omega} = FIM_{l,\omega}^{-1} = \frac{\sigma^2}{2} \times \begin{bmatrix} \tilde{G}^\dagger & \tilde{B} (D - CG B)^{-1} \tilde{C} \tilde{G}^{-1} \\ -(D - CG B)^{-1} \tilde{C} \tilde{G}^{-1} & (D - CG B)^{-1} \end{bmatrix}$$

Note that since we are only interested in parameter $\omega$, only the lower-right element of the matrix in (23) is needed and it is

$$\text{CRLB}_{\omega} = \frac{\sigma^2}{2} \times (D - CG B)^{-1} \tag{24}$$

where

$$\tilde{G} = \tilde{E}(I_{(K-1)\times(K-1)} - (1/K)I_{(K-1)\times(K-1)}) = \tilde{E}(I_{(K-1)\times(K-1)} - (1/K)I_{(K-1)\times(K-1)}) \tag{25}$$

Now to compute the $\tilde{G}^{-1}$, we can exploit one of the special cases of Binomial Inverse Theorem [17] to get

$$(Y + uu^T)^{-1} = Y^{-1} - \frac{Y^{-1}uu^TY^{-1}}{1 + u^TY^{-1}u} \tag{26}$$

where $Y$ is a square matrix and $u$ is a column vector. Applying the formula (26) on (25) assuming that $Y = I_{(K-1)\times(K-1)}$ and $u = I_{(K-1)\times1}$, we will have

$$\tilde{G}^{-1} = \frac{1}{E} (I_{(K-1)} + I_{(K-1)\times(K-1)}) \tag{27}$$

Thus,

$$\tilde{C} \tilde{G}^{-1} \tilde{B} = \text{Im}(p^H p) \tilde{L}^T \frac{1}{E} (I_{(K-1)} + I_{(K-1)\times(K-1)}) \text{Im}(p^H p) \tilde{L}$$

$$= \frac{1}{E} \left( \text{Im}(p^H p) \right)^2 \tilde{L}^T \frac{1}{E} \left( I_{(K-1)} + I_{(K-1)\times(K-1)} \right) \tilde{L}$$

$$= \frac{1}{E} \left( \text{Im}(p^H p) \right)^2 \left( \tilde{L}^T \tilde{L} + \tilde{L}^T l_{(K-1)\times(K-1)} \tilde{L} \right)$$

$$= \frac{1}{E} \left( \text{Im}(p^H p) \right)^2 \left( (L_2 - L_1^2 / K - L_1^2 / K^2) + (L_1^2 / K^2) \right)$$
Finally, by putting (28) in (24) and defining \( R_1 \triangleq (L_1 / K) \) and \( R_2 \triangleq (L_2 / K) \), we have

\[
CRLB_{ao} = \frac{\sigma^2}{2} \times (\bar{D} - \bar{C} \hat{G}^{-1} \hat{B})^{-1}
\]

\[
= \frac{\sigma^2}{2} \times [(L_2 - L_1^* / K)E_{0p} - \frac{1}{E} (\text{Im}(p^{*H} p))^2 (L_2 - L_1^*)]^{-1}
\]

\[
= \frac{K\sigma^2}{2} \times [(R_2 - R_1^*) (E_{0p} - \frac{C_0^2}{E})]^{-1}
\]

Thus, the CRLB for estimation of the Doppler-shifted frequency in coherent pulse train signal is

\[
CRLB_{ao} = \frac{\sigma^2}{2K(E_{0p} - \frac{C_0^2}{B_0 E_{0p}})(R_2 - R_1^*)}
\]

\[
= \frac{1}{2MK \text{SNR} \left( 1 - \frac{C_0^2}{B_0 E_{0p}} \right)} (R_2 - R_1^*)
\]

where \( \text{SNR} \triangleq \frac{E_{0p}}{M\sigma^2} \), \( K \) is the number of pulses, \( M \) is the number of samples per pulse, \( E_{0p} = \sum_{n=0}^{M-1} \left| p(n) \right|^2 \) is a measure of the pulse energy, \( B_0 = \left( \sum_{n=0}^{M-1} \left| p(n) \right|^2 \right) / \left( \sum_{n=0}^{M-1} \left| p(n+1) \right|^2 \right) \) is a measure of the pulse bandwidth, \( C_0 = \text{Im} \left( \sum_{n=0}^{M-1} p(n) p^{*} (n+1) \right) \) is a measure of time-frequency cross-coupling (measure of skew), \( R_t = \frac{1}{K} \sum_{k=1}^{K-1} l_k \) is the average of the pulse times, and \( R_2 = \frac{1}{K} \sum_{k=1}^{K-1} l_2 \) is a measure of the temporal spread of the pulse arrival times. Note that the term \( (R_2 - R_1^*) \) is the “mean-normalized” measure of the time spread of the pulse times (i.e. the variance of pulse times). These mean-normalized measures ensure that the choice of time origin has no impact on the CRLB value.

## III. Discussion and Conclusion

To allow some further insight into this result some special cases are now considered. When there is no phase/frequency modulation on the pulses, the \( \rho(t) \) will be purely real valued. Thus, the block matrices \( B, C, D \) in (20) will be block diagonal. In this special case, then, \( C_0 = 0 \) in (29) and the CRLB is then given by

\[
CRLB_{ao} = \frac{1}{2MK \text{SNR} (R_2 - R_1^*)}
\]

\[
= \frac{1}{2T_{or} F_s K \text{SNR} (R_2 - R_1^*)}
\]

where \( F_s \) is the sampling frequency and \( T_{or} = (M / F_s) \) is the Pulse On-Time, and we see that the CRLB varies as \( O(1/T_{or}) \). It may appear that it is possible to reduce the CRLB by increasing the sampling frequency \( F_s \), but this is not true since increasing the \( F_s \) leads to an SNR decrease and the decreased SNR perfectly counters the increased \( F_s \) so that the product \( \text{SNR} \times F_s \) remains almost the same.

In another case, when the intercepted pulses are equally spaced by \( T_{PRI} \), the pulse repetition interval (PRI), we have \( l_k = kT_{PRI} F_s \). Thus,

\[
(R_2 - R_1^*) = T_{PRI} F_s \sum_{k=1}^{K-1} k^2 \left( \frac{T_{PRI} F_s \sum_{k=1}^{K-1} k}{K} \right)^2
\]

\[
= \frac{T_{PRI}^2 F_s^2 (K^2 - 1)}{12}
\]

By putting (31) in (29), we have

\[
CRLB_{ao} = \frac{6}{MT_{PRI}^2 F_s^2 K (K^2 - 1) \text{SNR} \left( 1 - \frac{C_0^2}{B_0 E_{0p}} \right)} (R_2 - R_1^*)
\]

In this case the frequency CRLB varies as \( O(1/T_{PRI}^2) \) and as \( O(1/K^3) \).

To compare the result for the unknown pulse case to the known pulse case it is important to note that the general result in (29) and it variants are for frequency in units of rad/sample where as the results in [13] are for frequency in units of rad/sec. We can convert (29) into one for frequency in rad/sec by multiplying by \( F_s \) or equivalently dividing by \( \Delta^2 = 1/F_s^2 \) to give
\[ CRLB_{\text{unknown pulse}} = \frac{1}{2MK SNR \left( 1 - \frac{C_0^2}{B_0 E_{op}} \right) \Delta^2 (R_2 - R_1^2)} \]

where \( \Delta > 0 \) is as defined in (14) and (15) of [13] and is a measure of the duration of an individual pulse.

In terms of our notation the result in (13) of [13] can be written as

\[ CRLB_{\text{known pulse}} = \frac{1}{2MK SNR \left( 1 - \frac{C_0^2}{B_0 E_{op}} \right) \Delta^2 (\tilde{D} + R_2 - R_1^2)} \]

where \( \tilde{D} > 0 \) is as defined in (14) and (15) of [13] and \( \Delta \) is a measure of the duration of an individual pulse. Comparing these two results shows that the only difference is in the known-pulse result there is an extra term \( \tilde{D} \) that is added to \( R_2 - R_1^2 \) which reduces the CRLB. But note that in [13] it was discussed that in most real-world scenarios \( \tilde{D} \ll R_2 - R_1^2 \), but the effect of \( \tilde{D} \) becomes noticeable when the number of pulses is very small (essentially only when two pulses are available) and the pulse width is approximately equal to the PRI \( T_{\text{PRI}} \approx T_{\text{PRI}} \) which is not a common real-world scenario (especially given that even if \( T_{\text{PRI}} \approx T_{\text{PRI}} \), the number of pulses processed is usually much larger than two. Thus, regardless of pulse type (e.g., purely real or containing phase/frequency modulation) there is essentially no difference between the two results. This was an unexpected result – given the insights from [14] and [15] one would not expect lack of knowledge of a phase/frequency modulated pulse to have no impact on the CRLB. To ensure this was true (and not due to errors in developing the closed forms) the results were checked by forming the full FIMs and inverting them – the results matched the closed forms. Again, as pointed out in the Introduction, without the results provided here researchers and practitioners would expect the results in [13] to be overly optimistic – but that is actually not the case!

Numerical results are provided in Table I for several cases of SNR, pulse shape, pulse on-time and number of pulses; these results were computed using (33) and (34). For the results in Table I, the pulse repetition interval \( T_{\text{PRI}} \) is 1 ms, sampling frequency is \( F_s = 1 \) MHz and the pulse \( p(t) \) has a shape given by the Tukey window [18] with a linear frequency modulation from -100 kHz to 100 kHz. The column in Table I labeled \( \alpha \) is the taper parameter [18] that is used here to control the shape of the pulse (note that \( \alpha = 0 \) corresponds to zero taper which is a rectangular pulse; \( \alpha = 1 \) corresponds to full taper which is a Ham window), \( K \) is the number of pulses and \( T_{\text{PRI}} \) is the pulse on-time in \( \mu s \). The results for the known-pulse and unknown-pulse cases are remarkably similar. The results in Fig. 1 further illustrate the relationship between the known-pulse and unknown-pulse cases for two different PRI values as the pulse width approaches the PRI. The results show that the effect of \( \tilde{D} \) is largely swamped by \( R_2 - R_1^2 \) and only has a small effect when \( PW \approx PRI \) and the number of pulses is less than 4.

In conclusion, we were motivated to derive the CRLB for the unknown pulse case because previous work on other estimation problems had shown that assuming known signals or known phase gave overly optimistic values for the CRLB. Thus, we modeled the pulse as a deterministic unknown since we would not have any prior information about the transmitted signal. The results surprisingly show that the CRLB for the unknown-pulse case rapidly approaches that for the known pulse case when the number of pulses grows. However, two final points should be stressed that although knowledge of the pulse shape has little impact on the CRLB (i) the specific pulse shape itself does impact the CRLB and (ii) knowledge of the pulse shape (at least roughly) may be crucial to building a practical estimator.
In this appendix, we simplify (22) to drive the FIM and CRLB matrix of \( l \) and \( \omega \). Before that, to simplify the notation we define:

\[
E_{0p} \triangleq p^T M p = \sum_{n=0}^{M-1} |p(n\Delta)|^2 ;
\]

\[
E_{ip} \triangleq p^T M p = \sum_{n=0}^{M-1} |p(n\Delta)|^2 ;
\]

\[
E_{2p} \triangleq p^T M p = \sum_{n=0}^{M-1} n^2 |p(n\Delta)|^2 ;
\]

\[
L_2 \triangleq \sum_{k=1}^{K} l_k ; \quad L = \text{diag}(l_1, ..., l_{K-1});
\]

\[
\tilde{L} = [l_1 - L_1 / K \quad l_2 - L_2 / K \quad ... \quad l_{K-1} - L_1 / K]^T
\]

As mentioned in equation (17), it is easy to show that

\[
A^T N A = L_1 I + KM
\]  

where \( L_1 \triangleq \sum_{k=1}^{K-1} l_k \) and \( M \triangleq \text{diag}(0, 1, ..., M - 1) \). We will also use this equality in computing the elements of the matrix \( (D - CG^{-1}B) \) in (22). We can write the \( CG^{-1}B \) term in equation (22) as

\[
CG^{-1}B = \frac{1}{K} \times
\]

\[
\begin{bmatrix}
\tilde{P}_r^T AA^T \tilde{P}_r + \tilde{P}_r^T AA^T \tilde{P}_r & \tilde{P}_r^T AA^T \tilde{P}_r - \tilde{P}_r^T AA^T \tilde{P}_r & \tilde{P}_r^T AA^T \tilde{P}_r - \tilde{P}_r^T AA^T \tilde{P}_r \\\np_r^T A^\dagger N A \tilde{P}_r - p_r^T A^\dagger N A \tilde{P}_r & p_r^T (A^\dagger NA)^2 p_r + p_r^T (A^\dagger NA)^2 p_r
\end{bmatrix}
\]

\[
(37)
\]

Now, we simplify each elements of the matrix (38):

Using (9), (10) and (36), we have

\[
(\tilde{P}_r^T AA^T \tilde{P}_r + \tilde{P}_r^T AA^T \tilde{P}_r) / K = (\tilde{E}_{(K-1)(K-1)}) / K
\]  

(38)

Using (9), (10), (36) and (37), we have

\[
(\tilde{P}_r^T AA^T N A p_r - \tilde{P}_r^T AA^T N A p_r) / K =
\]

\[
= (\tilde{P}_r^T A(L_1 I_M + KM) p_r - \tilde{P}_r^T A(L_1 I_M + KM) p_r) / K
\]

\[
= [(L_1 / K) \tilde{P}_r^T A p_r + \tilde{P}_r^T AMP_r - (L_1 / K) \tilde{P}_r^T A p_r + \tilde{P}_r^T AMP_r]
\]

(39)
and,

\[(p_i^T A^T N A A^T \tilde{P}_r - p_i^T A^T N A A^T \tilde{P}_r)/K = (p_i^T (L_i I + KM) A^T \tilde{P}_r - p_i^T (L_i I + KM) A^T \tilde{P}_r)/K\]

(40)

and

\[(p_i^T (A^T N A)^2 p_i + p_i^T (A^T N A)^2 p_i)/K = (L_i^2 E_{0p} + 2L_i KE_{1p} + K^2 E_{2p})\]

As we see in the block matrix version of (21),

\[D \triangleq \begin{bmatrix} \tilde{E}(I_{K-1}) & \text{Im}(\tilde{P}^H N A p) \\ -\text{Im}(p^H A^T N \tilde{P}) & s^H N^2 s \end{bmatrix}\]

To more simplify the matrix \(D\), we have

\[
\text{Im}(\tilde{P}^H N A p) = (\tilde{P}_r^T N A p_r - \tilde{P}_r^T N A p_r) \\
= (L \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r) - (L \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r)
\]

(42)

\[-\text{Im}(p^H A^T N \tilde{P}) = (p^T A^T N \tilde{P}_r - p_i^T A^T N \tilde{P}_r) \]

\[= (p^T A^T \tilde{P}_r L^T + p_i^T MA^T \tilde{P}_r) - (p^T A^T \tilde{P}_r L^T + p_i^T MA^T \tilde{P}_r)
\]

(43)

where \(L = \text{diag}(l_1, \ldots, l_{K-1})\) and

\[s^H N^2 s = p^H A^T N^2 s p = p^H (L_2 I_M + 2L_1 M + KM^2) p \]

(44)

\[= L_2 E_{0p} + 2L_1 E_{1p} + KE_{2p}\]

Finally, using (46), (47) and (48), we have:

\[D - CG^{-1}B = \begin{bmatrix} \tilde{E}(I_{K-1}) - (1/K)I_{(K-1)(K-1)} & \text{Im}(p^H p) \tilde{L} \\
\text{Im}(p^H p) \tilde{L}^T & (L_2 - L_{12}^2 / K) E_{0p} \end{bmatrix}_{K \times K}
\]

(46)

Thus,

\[
\text{FIM}_{\text{ch}} = \frac{2}{\sigma^2} \times (D - CG^{-1}B) = \\
\frac{2}{\sigma^2} \begin{bmatrix} \tilde{E}(I_{K-1}) - (1/K)I_{(K-1)(K-1)} & \text{Im}(p^H p) \tilde{L} \\
\text{Im}(p^H p) \tilde{L}^T & (L_2 - L_{12}^2 / K) E_{0p} \end{bmatrix}_{K \times K}
\]

(48)

Now using the equations (36)-(45), we can find the term \((D - CG^{-1}B)\) in equation (22). To do that, we compute each one of the elements of the matrix \((D - CG^{-1}B)\) as following:

\[
\text{Im}(\tilde{P}^H N A p) - (\tilde{P}_r^T A^T N A p_r - \tilde{P}_r^T A^T N A p_r)/K = \\
= ((L \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r) - (L \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r)) \\
- \{(L_i / K) \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r - (L_i / K) \tilde{P}_r^T A p_r + \tilde{P}_r^T A M p_r\}
\]

\[= (L - (l_i / K)I_{(K-1)}) \tilde{P}_r A p_r - \tilde{P}_r A p_r) \]

\[= (p_i^T p_r - p_i^T p_r) \tilde{L}
\]

(45)

and,

\[-\text{Im}(p^H A^T N \tilde{P}) - (p_i^T A^T N A \tilde{P}_r - p_i^T A^T N A \tilde{P}_r)/K = \\
= (p_i^T p_r - p_i^T p_r) \tilde{L}
\]

(46)
REFERENCES


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