A Proof of the Schroder-Bernstein Theorem

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The following proof is a slightly modified version of C. A. Gunter and D. S. Scott's proof in their article *Semantic Domains* in Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics, pages 633–674, 1990.

Theorem 1 (Schroder-Bernstein) *Let S and T be sets. If $f : S \to T$ and $g : T \to S$ are injections, then there is a bijection $h : S \to T$.*

Proof. Let us for a moment assume that we can find a set $Y \subseteq T$ that satisfies the equation:

$$T \setminus Y = f^*(S \setminus g^*(Y)) \tag{1}$$

Define $h : S \to T$ by

$$h(x) = \begin{cases} y & \text{if } x \in g^*(Y) \text{, that is, } x = g(y) \text{ for some } y \in Y \\ f(x) & \text{if } x \in (S \setminus g^*(Y)) \end{cases}$$

To see that h is well defined for $x \in g^*(Y)$, notice that we have a unique choice of y because g is an injection.

To see that h is an injection, suppose $x_1, x_2 \in S$ and $h(x_1) = h(x_2)$. We have four cases. First, if $x_1, x_2 \in g^*(Y)$, then $x_1 = g(h(x_1)) = g(h(x_2)) = x_2$. Second, if $x_1, x_2 \in (S \setminus g^*(Y))$, then $x_1 = x_2$ because f is an injection. Third, if $x_1 \in g^*(Y)$ and $x_2 \in (S \setminus g^*(Y))$, then we have $h(x_1) \in Y$ and $h(x_2) \in f^*(S \setminus g^*(Y)) = (T \setminus Y)$, where the last step is equation (1). Now $h(x_1) \in Y$ and $h(x_2) \in (T \setminus Y)$ contradicts $h(x_1) = h(x_2)$. Fourth, if $x_1 \in (S \setminus g^*(Y))$ and $x_2 \in g^*(Y)$, then in a manner that is similar to the third case, we can reach a contradiction of $h(x_1) = h(x_2)$.

To see that h is a surjection, suppose $y \in T$. We have two cases. If $y \in Y$, then $h(g(y)) = y$. If $y \in (T \setminus Y)$, then we have from equation (1) that $y \in (T \setminus Y) = f^*(S \setminus g^*(Y))$, so $y = f(x) = h(x)$ for some $x \in (S \setminus g^*(Y))$.

We must finally prove that we can find a set $Y \subseteq T$ that satisfies the equation (1). The function $Y \mapsto (T \setminus f^*(S)) \cup f^*(g^*(Y))$ from subsets of T to subsets of T is easily seen to be continuous with respect to the inclusion ordering. Hence, by the Fixed Point Theorem, there is a subset $Y = (T \setminus f^*(S)) \cup f^*(g^*(Y))$. We have

$$\begin{aligned} T \setminus Y &= T \setminus [(T \setminus f^*(S)) \cup f^*(g^*(Y))] \\ &= f^*(S) \cap (T \setminus f^*(g^*(Y))) \\ &= f^*(S \setminus g^*(Y)) \end{aligned}$$

So our assumption is valid and we conclude that h is a bijection. $\square$