Efficient Recursive Subtyping

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Joint work with
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The Result

An efficient algorithm that decides if one recursive type is a subtype of another.

$$t ::= v \mid \bot \mid \top \mid t \to t \mid \mu v.t \qquad \text{(types)}$$

Amadio and Cardelli (POPL'91): Exponential time.

Kozen, Palsberg, and Schwartzbach (POPL'93): $O(n^2)$ time.

Outline of the talk:

1. Recursive types and subtyping
2. Amadio and Cardelli's algorithm
3. Our algorithm

Lambda Calculus

$$E ::= x \mid \lambda x.E \mid EE \qquad (\lambda\text{-terms})$$

$$t ::= v \mid \bot \mid \top \mid t \to t \mid \mu v.t \qquad (\text{types})$$

Typing rules:

$$A \vdash x : t \quad (A(x) = t)$$

$$\frac{A[x \leftarrow t_1] \vdash E : t_2}{A \vdash (\lambda x.E) : t_1 \to t_2}$$

$$\frac{A \vdash E_1 : t_1 \to t_2 \quad A \vdash E_2 : t_1}{A \vdash (E_1 E_2) : t_2}$$

$$\frac{A \vdash E : t \quad t \leq t'}{A \vdash E : t'}$$

Recursive Types

Motivation: typing recursive functions and data structures.

A recursive type $\mu v.t$ satisfies

$$\mu v.t = t[v/\mu v.t]$$

Deciding type equality: $O(n\alpha(n))$ time.

Subtyping

Motivation: typing functions in object-oriented languages.

Type ordering:

$t \leq \top$ for any type t
$\perp \leq t$ for any type t

Add the usual rule for ordering function spaces:

$s \to t \leq s' \to t'$ if and only if $s' \leq s$ and $t \leq t'$

How can we order recursive types?

Recursive Types + Subtyping

Found in Amber and Quest.

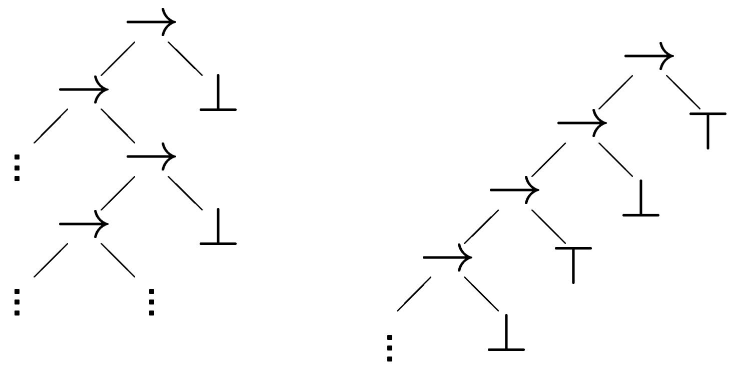
$$t ::= v \mid \bot \mid \top \mid t \to t \mid \mu v.t \qquad \text{(types)}$$

Studied by Amadio and Cardelli [POPL'91].

$$(v \leq v' \Rightarrow t \leq t') \Rightarrow (\mu v.t \leq \mu v'.t')$$

Representing types as labeled trees:

$$\mu u.((u \to u) \to \bot) \;\leq\; \mu v.((v \to \bot) \to \top)$$



Amadio and Cardelli's Algorithm

Problem: Given two types s and t, is $s \leq t$?

Definition
Two trees are ordered if all their finite truncations are ordered.

Concrete representation of trees involving back-pointers.

Eksplicit construction of a binary proof tree of polynomial depth.

Time complexity: exponential.

Our Algorithm

Definition
Two trees are ordered if no common path detects a counterexample.

Theorem
Our definition is equivalent to Amadio and Cardelli's.

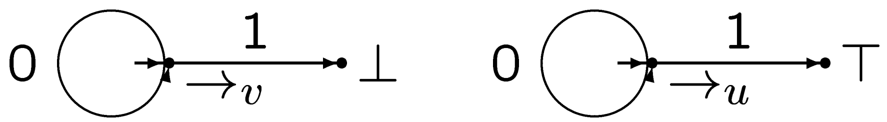
Representation of trees as *term automata*.

A certain product automaton accepts all counterexamples.

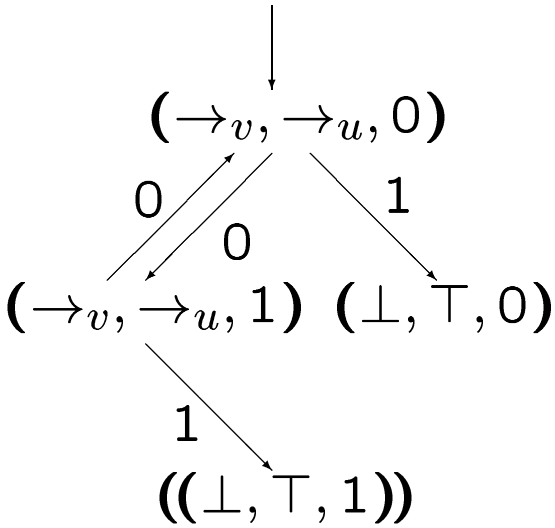
Time complexity: $O(n^2)$.

Example

$$\mu v.(v \to \bot) \quad \not\leq \quad \mu u.(u \to \top)$$



Product automaton:



Ordering: $\bot \leq \to \leq \top$.

Perhaps reverse? 0 means "no", 1 means "yes".

Terms

Intuition: a term is a labeled tree.

Σ, a finite ranked alphabet.

In our application: $\Sigma = \{\bot, \rightarrow, \top\}$

A term over Σ is a partial function

$$t : \omega\star \ \rightarrow \ \Sigma$$

with domain $\mathcal{D}(t)$ satisfying

- $\mathcal{D}(t)$ is nonempty and prefix-closed;
- if $t(\alpha) \in \Sigma_n$,
  then $\{i \mid \alpha i \in \mathcal{D}(t)\} = \{0, 1, \ldots, n-1\}$.

The set of all terms over Σ is denoted T_Σ.

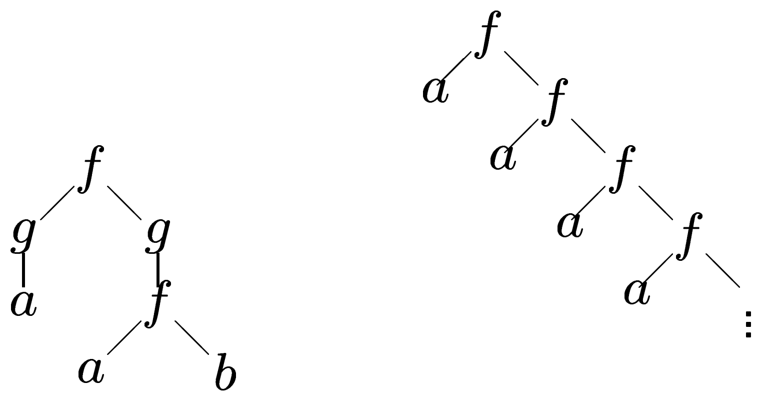
A term t is *finite* if its domain $\mathcal{D}(t)$ is a finite set.
We denote the set of finite terms over Σ by F_Σ.

Subterms

Subterms: $t{\downarrow}\alpha(\beta) = t(\alpha\beta)$

A term t is *regular* if it has only finitely many distinct subterms; *i.e.*, if $\{t{\downarrow}\alpha \mid \alpha \in \omega\star\}$ is a finite set.

Example terms:



If t is a term, then the following are equivalent:

(i) t is regular;

(ii) t is representable by a term automaton;

(iii) t is described by a finite set of equations involving the μ operator.

Our Type Ordering

Types are terms over $\Sigma = \{\bot, \to, \top\}$.
Over this signature, every $\mathcal{D}(t) \subseteq \{0, 1\}\star$.

The order $\leq_{\mathsf{FIN}}$ is the smallest binary relation on F_Σ such that

(i) $\bot \leq_{\mathsf{FIN}} t \leq_{\mathsf{FIN}} \top$ for all finite t;

(ii) if $s' \leq_{\mathsf{FIN}} s$ and $t \leq_{\mathsf{FIN}} t'$ then $s \to t \leq_{\mathsf{FIN}} s' \to t'$.

The *parity* of $\alpha \in \{0, 1\}\star$ is the number mod 2 of 0's in α. The parity of α is denoted $\pi\alpha$. A string α is said to be *even* if $\pi\alpha = 0$ and *odd* if $\pi\alpha = 1$.

Let $\leq_0$ be the linear order

$$\bot \ \leq_0 \ \to \ \leq_0 \ \top$$

on Σ, and let $\leq_1$ be its reverse

$$\top \ \leq_1 \ \to \ \leq_1 \ \bot \ .$$

For $s, t \in T_\Sigma$, define $s \leq t$ if $s(\alpha) \leq_{\pi\alpha} t(\alpha)$ for all $\alpha \in \mathcal{D}(s) \cap \mathcal{D}(t)$.

Basic Result

The relation $\leq$ is a partial order on T_Σ, and agrees with $\leq_{\mathsf{FIN}}$ on F_Σ. In particular, for any s, t, s', t',

(i) $\bot \leq t \leq \top$

(ii) $t \leq \bot$ if and only if $t = \bot$

(iii) $\top \leq t$ if and only if $t = \top$

(iv) $s \to t \leq s' \to t'$ if and only if $s' \leq s$ and $t \leq t'$.

Recall that $s \not\leq t$ iff
$\{\alpha \in \mathcal{D}(s) \cap \mathcal{D}(t) \mid s(\alpha) \not\leq_{\pi\alpha} t(\alpha)\}$ is nonempty.

Theorem
A certain product automaton accepts

$$\{\alpha \in \mathcal{D}(s) \cap \mathcal{D}(t) \mid s(\alpha) \not\leq_{\pi\alpha} t(\alpha)\}$$

Algorithm

Input: Two types s and t

Output: Is $s \leq t$?

1. Construct the term automata for s and t (linear time).

2. Construct the product automaton as sketched before (size $O(n^2)$).

3. Decide, using depth first search, if the product automaton accepts the empty set (linear time).

4. If so then $s \leq t$ else $s \not\leq t$.

Time complexity: $O(n^2)$.

Tree Truncations

The *level-k truncations* $t|_k$, $t\lceil_k$, and $t\lfloor_k$ of the term t are defined by

$$\mathcal{D}(t|_k) = \mathcal{D}(t\lceil_k) = \mathcal{D}(t\lfloor_k) = \{\alpha \in \mathcal{D}(t) \mid |\alpha| \leq k\}$$

and

$$\begin{aligned}
t|_k(\alpha) &= \begin{cases} t(\alpha), & |\alpha| < k \\ \top, & |\alpha| = k \end{cases} \\
t\lceil_k(\alpha) &= \begin{cases} t(\alpha), & |\alpha| < k \\ \lceil\alpha\rceil, & |\alpha| = k \end{cases} \\
t\lfloor_k(\alpha) &= \begin{cases} t(\alpha), & |\alpha| < k \\ \lfloor\alpha\rfloor, & |\alpha| = k \end{cases}
\end{aligned}$$

where

$$\lceil\alpha\rceil \;=\; \begin{cases} \bot, & \alpha \text{ even} \\ \top, & \alpha \text{ odd,} \end{cases} \qquad \lfloor\alpha\rfloor \;=\; \begin{cases} \top, & \alpha \text{ even} \\ \bot, & \alpha \text{ odd.} \end{cases}$$

Notice that $t\lceil_k \leq_{\mathsf{FIN}} t\lceil_{k+1}$

Main Result

Lemma:
For any $k \geq 0$ and terms s and t, the following are equivalent:

- $s|_k \ \leq_{\mathsf{FIN}} \ t|_k$
- $s\lceil_k \leq_{\mathsf{FIN}} \ t\lceil_k$
- $s\lfloor_k \leq_{\mathsf{FIN}} \ t\lfloor_k$.

Definition:
Amadio-Cardelli's order $\leq_{\mathsf{AC}}$ is defined as

$$s \leq_{\mathsf{AC}} t \iff s|_k \leq_{\mathsf{FIN}} t|_k \text{ for all } k \geq 0 \ .$$

Theorem: The relation $\leq$ agrees with $\leq_{\mathsf{AC}}$.

Conclusion

- Automata!
- Fast algorithms for type systems based on contravariance.

Future work
Extend the algorithms to handle more constant types.