Solving the Live-out Iterator Problem, Part I

Louis-Noël Pouchet

pouchet@cse.ohio-state.edu

Dept. of Computer Science and Engineering, the Ohio State University

September 2010

888.11

Reminder: step-by-step methodology

1. $\surd$ **Problem definition:** Understand and define the problem
2. $\surd$ **Examples:** Find various example, and compute the desired output by hand
3. $\longrightarrow$ **Restriction:** Find an algorithm, maybe restricted to simpler cases
4. **Generalization:** Generalize the algorithm to work on all cases
5. **Proof:** Prove the algorithm is complete and correct
6. **Complexity:** Study the complexity of the algorithm

Outline for today

- Find a useful restriction of the problem
  - Typically, add extra properties on the input
  - And/or remove some properties on the output
- Build and solve the problem for it
  - Maximal reuse of existing solutions
  - Keep in mind the general problem

Summary of the problems

List the problems to solve

- Multiple statements
  - Which loop executes last?
- Min/max expressions
  - The value depends on the expressions
  - Need to substitute surrounding iterators with the last value for which the loop test is true, not necessarily the exit value of the loop iterator
- Conditionals
  - a loop may not execute, how to determine its last execution?
- Parametric loop bounds
  - The loop may not execute at all!
  - What is the value to use in the substitution? The exit value?
- Loop iterator symbols being assigned after the loop execution
  - How to compute the exit value in this case?

Another view: Solution-driven

Order the problems starting with the simplest solution

1. Start from the set of programs with:
   - no conditional,
   - no min/max,
   - no parameter,
   - no iterator symbol assigned in the loop body,
   - a single statement
2. Adding multiple statement support
3. **Adding parameters**
4. Adding conditionals
5. Adding min/max
6. Adding iterator symbol assigned in the loop body

A useful restriction of the problem

- What if a loop always iterates at least once?
  - Property: $lb \leq Ub$
  - The exit value is the last value for which the test is true + 1
  - Impact on conditionals, min/max, iterator assigned in body?
- What if a conditional is always true?
  - Property: the conditional is an affine form of the parameters only
- Under these assumptions, what about min/max expressions?

Overview of the approach

1. Find a good, general algorithm for our restricted case
2. Modify it to generalize to:
   - arbitrary conditionals
   - arbitrary loop bounds
3. Modify the input specification to cover only programs where iterator symbols are never assigned outside the loop

Reminder: algorithm writing 101

1. Determine the input and output
2. Find a correct data structure to represent the problem
   - Don't hesitate to convert the input to a suitable form, and to preprocess it
3. Try to reduce your problem to a variation of a well-known one
   - Sorting? Path discovery/reachability? etc.
   - Look in the litterature if a solution to this problem exists
4. Decide wheter you look for a recursive algorithm or an imperative one, or a mix
   - Depends on how you think, how easy it is to exhibit invariants, what is the decomposition in sub-problems, ...
5. Write the algorithm :-)
6. Run all your examples on it, manually, before trying to prove it

Determine the input and output

Input:

- an AST A of a program such that:
  - A represents a Static Control Part
  - For each loop in A, the lower bound is always smaller than the upper bound
  - Conditionals are always true
  - There is no loop iterator symbol assigned outside its defining loop

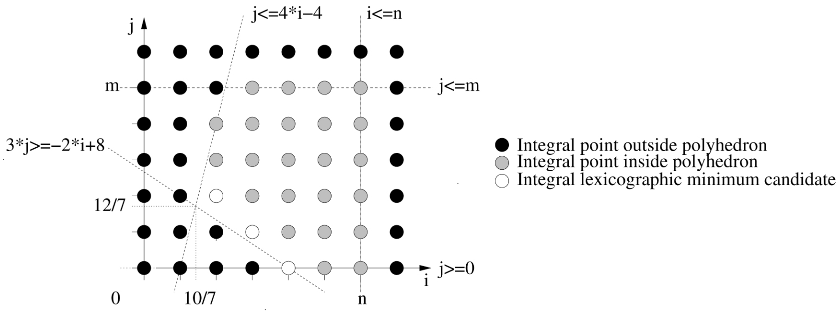
Output:

- an AST B containing A which is appended another AST that assigns to each loop iterator in A the value it takes when A is executed

Find a good representation for the problem

Example

```
for (i = 0; i <= N; ++i)
  for (j = 0; j <= min(M, 4 * i - 4); ++j)
    if (3 * j >= - 2 * i + 8)
      S(i, j);
```



Polyhedral representation

- Model iteration domains using inequalities
  - inequalities for lower bounds, upper bounds, conditionals
  - min/max simply produces multiple inequalities
  - Warning: only **executed** instances are part of the iteration domain
- Using this representation, what is the geometric intuition of the exit value of iterators?
  - It is simply the lexicographic maximum of the iteration domain + 1!
  - Can we reuse existing algorithms to compute the lexicographic maximum of the iteration domain?

Reducing to a variation of a well-known problem

PIP: Parametric Integer Programming [Fea88]

In a nutshell:

- PIP input: A system of inequalities defining a parametric polyhedron

  Example: $$\begin{cases} i \geq 0 \\ i \leq N \\ j \geq 0 \\ j \leq M \\ j \leq 4*i-4 \\ 3*j \geq -2*i+8 \end{cases}$$

- PIP output: the lexicographic **minimum** of the system

  Example:

```
if (7*n >= 10) {
  if (7*m >= 12)
    (i = 2, j = 2)
  if (2*n+3*m >= 8)
    (i = -m-(m div 2)+4, j = m)
}
```

Problems to solve

- PIP outputs the lexicographic minimum, we want the maximum
  - Simple: $max(x) = min(-x)$
  - Need to insert variables $x\prime = -x$, $y\prime = -y$, etc. as the first variables of the system, and compute the lexmin of the new system
- PIP does not produce an AST explicitly, it uses its internal representation
  - Need to convert PIPLib internal representation into an AST
  - Need to dig into PIPLib documentation, should not be difficult

On the road to write the algorithm

In a nutshell:

1. Convert the AST into its polyhedral representation
2. For a given statement, create the PIP problem for the lexmax
3. Convert the solution to the system into an AST

Data structures [1/2]

Polyhedral representation:

- It is a array of elements of type $Statement$
- A $Statement$ is a structure containing:
  - $Matrix : domain$, for the iteration domain, using the same representation as PIP input
  - $Matrix : schedule$, for the schedule
  - $integer : nbIter$, for the number of loops surrounding the statement
  - (and more, but not useful here)

- Available functions:
  - $Statement[] : extractPolyhedralRepresentation(AST : A)$
  - $Statement[] : orderInExecutionOrder(Statement[] : statementarray)$

Data structures [2/2]

PIP / PIPLib:

- PIPLib uses as an input a $Matrix$
- Calling PIPLib outputs a $QUAST$ (quasi-affine solution tree)
  - It is a tree where the leaves are all possible values for the lexicographic minimum of the input system, the other nodes are conditions on parameters

- Available functions:
  - $QUAST : computeLexicographicMinimum(Matrix : system)$
  - $AST : convertQuastToAST(QUAST : solution)$

Exercise

Input:

- an AST A of a program such that:
  - A represents a Static Control Part
  - For each loop in A, the lower bound is always smaller than the upper bound
  - Conditionals are always true
  - There is no loop iterator symbol assigned outside its defining loop

Output:

- an AST B containing A which is appended another AST that assigns to each loop iterator in A the value it takes when A is executed

Exercise: write an algorithm which implements the above description

Algorithm to create a Lexmax system

Algorithm

Algorithm extendSystemForLexmax
Input:
Matrix: A, in PIPLib format
integer: $nbVars$
Output:
Matrix: in PIPLib format, with extra columns and equalities such that lexmin(B) = lexmax(A) for the nbVars first variables

B ← *duplicateMatrix(A)*
for *i* ← *1* **to** *nbVars* **do**
  B ← *insertColumnAtPosition(B, 1)*
end for
for *i* ← *1* **to** *nbVars* **do**
  B ← *insertRowAtPosition(B, B.NbRows)*
  B[B.NbRows - 1][i] ←*-1*
  B[B.NbRows - 1][i + nbVars] ← *1*
end for
return *B*