A simple mathematical explanation of “Under the ruler, faster than the ruler”

June 8, 2010

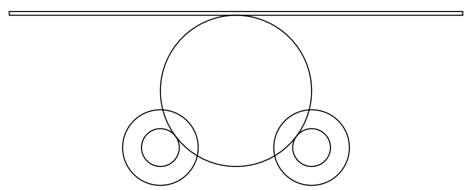

Illustration of wheels, under the ruler

Suppose the ruler is moving at a velocity v to the right.

Let a be the radius of the big center wheel. Let α be the angular velocity of the big wheel turning anti-clockwise.

Let b be the radius of the **inside** of the small wheels, where they touch the big wheel. Let c be the radius of the **outside** of the small wheels, where they touch the floor. Let β be the angular velocity of the smaller wheels turning clockwise.

Then, we have the following equation:

Since the big wheel and the small wheels touch each other and do not slip, we have that:

$$a \cdot \alpha = b \cdot \beta$$

Now, let's think of the velocity of the cart in two different ways. First, looking at the top of the cart, we see that the velocity of the cart must equal:

$$v + a \cdot \alpha$$

But looking at the bottom wheels, we also know that the velocity of the cart is:

$$c \cdot \beta$$

Thus, we have:

$$v + a \cdot \alpha = v + b \cdot \beta = c \cdot \beta$$

And so:

$$v = \beta(c - b)$$

What this means is that if $c = b$, then v must be 0, and so if you move the ruler at all, either slippage would occur or the cart would topple over. Also, if you imagine a situation that allowed $c < b$ (a weirdly shaped wheel whose ends sit on rails suspended in the air), then this would allow the cart to move to the left if the ruler moved to the right.

Finally, the velocity of the cart is:

$$\beta \cdot c = v \left(\frac{c}{c - b} \right)$$

So if $b < c$, then indeed $\frac{c}{c-b} > 1$, and so the cart moves under the ruler, faster than the ruler...

Amit Sahai