Problem Set II

1. We proved in class that DISJ has unbounded-error communication complexity $O(\log n)$. Prove that this bound is asymptotically tight. You must solve this problem from first principles, without appeal to the techniques of Lecture 15 such as Forster’s method.

2. Determine the unbounded-error communication complexity of EQ, up to a multiplicative constant.

3. A Euclidean embedding of $A \in \{-1,+1\}^{n \times m}$ is a system of vectors $u_1, \ldots, u_n, v_1, \ldots, v_m \in \mathbb{R}^k$ (for finite but arbitrarily large $k$) such that $A_{ij} = \text{sgn}(u_i, v_j)$ for all $i, j$. The margin of this Euclidean embedding is defined as $\min_{i,j} |\langle u_i, v_j \rangle|$. The margin of $A$, denoted $m(A)$, is the supremum of the margin over all Euclidean embeddings of $A$. Prove that $c \leq \text{disc}(A)/m(A) \leq C$ for some absolute constants $C > c > 0$ and all $A \in \{-1,+1\}^{n \times m}$.

4. A village has $n$ residents. Every day at noon, they all meet at the main square to discuss daily matters. An evil spirit visits the village one night and marks the ears of $k$ of the villagers with indelible ink, $k \geq 1$. Later that day, at noon, the evil spirit comes to the village meeting and announces that at least one villager has a marked ear. The evil spirit never visits the village again. If (and only if) a villager is able to logically deduce that he has a marked ear, he will leave the village the same day, never to be seen again. Ear marks being a taboo subject in town, the villagers never discuss it in any way. Moreover, a villager with a marked ear can never see his own mark in a mirror or otherwise. What will be the population count in the village $n+1$ days after the evil spirit’s visit?

5. Prove that the containment $\Sigma_1^{cc} \cup \Pi_1^{cc} \subset \Sigma_2^{cc} \cup \Pi_2^{cc}$ is strict.

6. Define $E(n,d) = \min p \sum_{x \in \{0,1\}^n} |p(x_1, x_2, \ldots, x_n) - x_1 x_2 \cdots x_n|$, where the minimum is over all polynomials $p$ of degree at most $d$. In words, $E(n,d)$ is the least $\ell_1$ error in an approximation of the AND function by a polynomial of degree at most $d$. Determine $E(n,d)$ for $d = 0, 1, 2, \ldots, n$.

7. Let $N_1, N_2, \ldots, N_k$ be norms on $\mathbb{R}^n$. Consider the norm $N$ given by $N = \max(N_1, N_2, \ldots, N_k)$. Prove that $B_N = \cap B_{N_i}$ and $B_{N^*} = \text{conv}(\cup B_{N_i})$.

8. Complete the proof of Forster’s theorem by showing that $\text{range}(\sum_{x \in X} xx^T) = \text{span} X$ for any finite $X \subset \mathbb{R}^n$.

9. A Boolean formula in Boolean variables $z_1, \ldots, z_n$ is a fully parenthesized Boolean expression with literals $z_1, \neg z_1, \ldots, z_n, \neg z_n$ and binary operators $\land$ and $\lor$. Formally, a Boolean formula is recursively defined as follows: each of $z_i$ and $\neg z_i$ is a Boolean formula; if $F$ and $G$ are Boolean formulas, then so are $(F) \land (G)$ and $(F) \lor (G)$. Let $F(z_1, \ldots, z_n)$ be a Boolean formula in which every variable occurs exactly once. Determine the deterministic communication complexity of $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$ given by $f(x,y) = F(x_1 \oplus y_1, \ldots, x_n \oplus y_n)$.

10. For a random $A \in \{-1,+1\}^{n \times n}$, prove that $\mathbf{P} \left[ \|A - E\| \geq t \right] \leq \exp(-\Omega(t^2))$. 