University of California, Los Angeles
CS 289A Communication Complexity
Instructor: Alexander Sherstov

Date assigned: February 13, 2012
Date due: February 22, 2012

Problem Set I

1. Prove that for 99% of communication problems $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, the characteristic matrix has rank $2^n - O(1)$ over the reals. In particular, the rank technique gives tight lower bounds on the deterministic communication complexity of random functions.

2. Prove or disprove: for every integer $k \geqslant 1$, the complexity class Σ_k^{cc} has a complete problem.

3. Consider the following public-coin randomized protocol on $\{0,1\}^n \times \{0,1\}^n$: Alice and Bob pick a uniformly random function $f\colon \{0,1\}^n \to \{0,1\}$ using shared randomness, exchange the values $f(x)$ and $f(y)$, and output 1 if and only if $f(x) = f(y)$. Prove that this is a protocol for the equality problem EQ_n with one-sided error 1/2. Adapt it to reduce the error to any given $\varepsilon > 0$ without repetition. How much communication does the new protocol require, as a function of ε?

4. Let $A \in \mathbb{R}^{n\times m}$ be a given matrix. Prove that there exists a vector $x > 0$ with $Ax = 0$ if and only if there does not exist a vector y with $y^{\mathsf{T}}A \geqslant 0$ and $y^{\mathsf{T}}A \neq 0$. (When applied to vectors, inequalities are interpreted componentwise, e.g., $x > 0$ means that $x_i > 0$ for all i).

5. Let M be a matrix with nonnegative entries. Define the *positive rank* of M, denoted $\mathrm{rk}_+ M$, to be the least k for which $M = M_1 + M_2 + \cdots + M_k$, where each M_i is a rank-1 matrix with nonnegative entries. Assuming that $\mathrm{rk}_+ M \leqslant \exp((\log_2 \mathrm{rk}_{\mathbb{R}} M)^c)$ for some constant $c > 1$ and every $M \in \{0,1\}^{n\times m}$, prove the log-rank conjecture.

6. For linear subspaces $A, B \subseteq \mathbb{F}_2^n$, define $f(A,B) = 1$ if and only if A and B are orthogonal. Prove that $N(f) = \Theta(n^2)$.

Bonus problem:

Prove that every communication problem $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ has a public-coin randomized protocol with cost 2 bits and error at most $\frac{1}{2} - \Omega(2^{-n/2})$. *Hint:* prove first that for any fixed $u \in \mathbb{R}^N$ and a uniformly random $z \in \{-1,+1\}^N$, one has $\mathbb{E}_z[|\langle z,u\rangle|] \geqslant \Omega(\|u\|_1/\sqrt{N})$.