You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

(3 pts) 1 Give a simple verbal description of the language recognized by the following DFA.

Solution. All binary strings that contain 0010.

2 Draw NFAs for the following languages, taking full advantage of nondeterminism:

(2 pts) a. strings over alphabet \{0, 1, \ldots , 9\} where the final digit has appeared before;
(2 pts) b. binary strings in which there is a pair of 0s separated by a number of positions that is a multiple of 4.
Prove that the following languages over the binary alphabet are regular:

(2 pts) a. strings in which the number of 0s and the number of 1s are both even;
(2 pts) b. strings with at most one occurrence of the substring 00 (the string 000 has two);
(2 pts) c. strings in which the 1000th symbol from the end is a 1.

**Solution.**

a. The language is recognized by the following DFA:

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0 1
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b. The complement of this language is recognized by the following NFA and is therefore regular:

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0, 1
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Since regular languages are closed under complement, the original language is regular as well.

c. To recognize this language, a DFA simply needs to keep track of the last 1000 symbols seen, and accept if and only if there is a 1 in position 1000. Formally, the language is recognized by the DFA

\[ \delta(w_1 w_2 w_3 \ldots w_{1000}, \sigma) = w_2 w_3 \ldots w_{1000} \sigma. \]
Let $L$ be a regular language over the binary alphabet. Consider the following language over the same alphabet: $L' = \{ w : |w| = |u| \text{ for some } u \in L \}$. Prove that $L'$ is regular.

**Solution.** To obtain an NFA for $L'$, start with a DFA for $L$ and change all edge labels to “0, 1”.

(3 pts) 5 Prove that at most $k^{2k+1}2^k$ languages over the binary alphabet can be recognized by a DFA with $k$ states.

**Solution.** Simply count the number of distinct DFAs with $k$ states. Name the states $1, 2, 3, \ldots, k$. Then a DFA is a tuple

$$(\{1, 2, 3, \ldots, k\}, \{0, 1\}, \delta, q_0, F).$$

where

$q_0 \in \{1, 2, 3, \ldots, k\}$
$F \subseteq \{1, 2, 3, \ldots, k\}$,
$\delta : \{1, 2, 3, \ldots, k\} \times \{0, 1\} \to \{1, 2, 3, \ldots, k\}$.

Thus, the number of distinct ways to choose $(q_0, F, \delta)$ is

$$k \times 2^k \times k^{2k}.$$
For a language $L \subseteq \Sigma^*$, define $\text{insert}(L) = \{u \sigma v : u \in L, \sigma \in \Sigma\}$. Thus, $\text{insert}(L)$ is the set of all strings obtained by taking a string in $L$ and inserting a new character at some position. Prove that if $L$ is regular, so is $\text{insert}(L)$.

**Solution:**

For a language $L \subseteq \Sigma^*$, define $\text{suffix}(L) = \{v : uv \in L \text{ for some } u\}$. Thus, $\text{suffix}(L)$ is the set of all suffixes of strings in $L$. Use the closure of regular languages under the reverse and prefix operations to prove that $\text{suffix}(L)$ is regular whenever $L$ is regular.

**Solution.** To generate the suffixes of all strings in $L$, one can reverse the strings in $L$, generate all prefixes in the resulting language, and finally reverse the resulting strings. Thus,

$$\text{suffix}(L) = \text{reverse}(\text{prefix}(\text{reverse}(L))).$$

Since $L$ is regular and regular languages are closed under the prefix and reverse operations, it follows that $\text{suffix}(L)$ is regular.