You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Give a simple verbal description of the language recognized by the following DFA.

![DFA Diagram]

**Solution.** All binary strings that end with 0010.

2. Draw NFAs for the following languages, taking full advantage of nondeterminism:

   a. strings over alphabet \{0, 1, \ldots, 9\} where the final digit has *not* appeared before;
   b. binary strings that begin or end with 00 or 11.

**Solution.**

![NFA Diagrams]

a.  

b.  

Prove that the following languages over the binary alphabet are regular:

(a) strings in which every 1 is immediately followed by 00; (2 pts)
(b) strings that contain both 010 and 101 as substrings; (2 pts)
(c) strings that end with 00. (2 pts)

Solution.

(a) The language is recognized by the following NFA:

![NFA for (a)](image)

(b) The language of strings containing 010 is recognized by

![NFA for (b)](image)

Analogously, the language of strings containing 101 is recognized by

![NFA for (c)](image)

Since regular languages are closed under intersection, the language in the problem statement is regular.

(c) The language is recognized by the following NFA:

![NFA for (c)](image)
4. Let $L_1$ and $L_2$ be languages that can be recognized by NFAs with $n_1$ and $n_2$ states, respectively. Prove that $L_1 \cap L_2$ can be recognized by a DFA with at most $2^{n_1+n_2}$ states.

**Solution.** Using the method discussed in class, transform the NFAs for $L_1$ and $L_2$ to DFAs with at most $2^{n_1}$ and $2^{n_2}$ states, respectively. Then combine these DFAs to obtain a DFA for $L_1 \cap L_2$ with at most $2^{n_1} \times 2^{n_2} = 2^{n_1+n_2}$ states, using the Cartesian-product construction from class.

5. Prove that every language $L \subseteq \{0, 1\}^*$ can be expressed as a countable union $L = \bigcup_{i=1}^{\infty} R_i$ for some regular languages $R_1, R_2, R_3, \ldots$.

**Solution.** For any string $w \in \{0, 1\}^*$, the language $\{w\}$ is regular because it is recognized by the NFA

Now define $R_i$ to be the language whose only string is the $i$th string in $L$, in lexicographic order (with the understanding that $R_i = \emptyset$ if $L$ has fewer than $i$ strings). Then

$$L = \bigcup_{i=1}^{\infty} R_i,$$

where each $R_i$ is regular by above.
For a language \( L \subseteq \Sigma^* \), define \( \text{delete}(L) = \{ uv : u \sigma v \in L \text{ for some } \sigma \in \Sigma \} \). Thus, \( \text{delete}(L) \) is the set of all strings obtained by taking a nonempty string in \( L \) and deleting a single character from it. Prove that if \( L \) is regular, so is \( \text{delete}(L) \).

**Solution:**

![Diagram of DFA and NFA](image)

For a language \( L \), define \( \text{suffix}(L) = \{ v : uv \in L \text{ for some } u \} \). Thus, \( \text{suffix}(L) \) is the set of all suffixes of strings in \( L \). Show how to transform a DFA for any given regular language \( L \) into an NFA for \( \text{suffix}(L) \).

**Solution.** Add \( \epsilon \)-transitions from the initial state \( q_0 \) to every state reachable from \( q_0 \).