You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Prove that the following languages are regular:

   (2 pts) a. binary strings that contain neither 010101 nor 000111 as substrings;
   (2 pts) b. nonempty binary strings in which the first and last symbols are different;
   (2 pts) c. strings over the decimal alphabet \{0, 1, 2, \ldots, 9\} with symbols in sorted order.

**Solution.** The languages in (a)–(c) are given by

\[
\{0, 1\}^*\{010101\}\{0, 1\}^* \cup \{0, 1\}^*\{000111\}\{0, 1\}^*, \\
\{0\}\{0, 1\}^*\{1\} \cup \{1\}\{0, 1\}^*\{0\}, \\
\{0\}^*\{1\}^* \cdots \{9\}^*,
\]

respectively. Thus, they are obtained by applying the operations of Kleene star, concatenation, union, and complement to the regular languages \{0\}, \{1\}, \ldots, \{9\}, \{0, 1\}, \{010101\}, \{000111\}. Since regular languages are closed under these operations, the resulting languages are regular as well.
2 Give a simple verbal description of the language recognized by the following DFA.

Solution. Binary strings in which the number of 0s is a multiple of 5.

3 Draw NFAs for the following languages, taking full advantage of nondeterminism:

(2 pts) a. binary strings that contain a pair of 0s separated by two or more symbols;
(2 pts) b. even-length binary strings in which symbols come in pairs, e.g., ε, 00, 0011, 1111, but not 0, 01, 001101.
Let $L$ be a regular language over a given alphabet $\Sigma$. Define $L'$ to be the set of all strings in $L$ whose length is not a multiple of 2015. Prove that $L'$ is regular.

**Solution.** The language, call it $A$, of strings whose length is not a multiple of 2015 is regular, with DFA $(\{0, 1, 2, \ldots, 2014\}, \Sigma, \delta, 0, \{1, 2, \ldots, 2014\})$ where $\delta(q, \sigma) = (q + 1) \mod 2015$. Since $L' = L \cap A$ and regular languages are closed under intersection, it follows that $L'$ is regular.

Consider the language $L$ whose strings are binary encodings (with leading zeroes ignored) of multiples of 17. Thus, $L$ contains $\epsilon, 0, 00, 10001, 000010001, 110011$ but not $10, 11, 0001$. Prove that $L$ is regular.

**Solution.** We will give a DFA for $L$. The idea is to convert the input string on the fly to a decimal number and check to see if that number is a multiple of 17. A direct implementation of this idea results in the “infinite” automaton $(\{0, 1, 2, \ldots\}, \{0, 1\}, \delta, 0, \{0, 17, 34, \ldots\})$, where $\delta(q, \sigma) = 2q + \sigma$. The only flaw in this construction is that it uses infinitely many states.

To get an actual DFA for $L$, recall that we only care about the remainder of the decimal number modulo 17. Therefore, we can perform all intermediate computations modulo 17. This corresponds to the automaton $(\{0, 1, 2, \ldots, 16\}, \{0, 1\}, \delta, 0, \{0\})$, where $\delta(q, \sigma) = (2q + \sigma) \mod 17$. 
For a language \( L \) over the binary alphabet, let \( \text{edit}(L) \) denote the set of strings that can be obtained from a string in \( L \) by flipping exactly one bit. Prove that \( \text{edit}(L) \) is regular whenever \( L \) is regular.

**Solution.** A DFA for \( L \) can be transformed into an NFA for \( \text{edit}(L) \) as follows, where \( \delta \) refers to the DFA’s transition function.

Let \( L \) be the language of strings over alphabet \( \{a, b, c\} \) in which the number of \( a \)’s, the number of \( b \)’s, and the number of \( c \)’s are either all even, or precisely one of them is even. Give a DFA for \( L \) with at most four states.

**Solution.**