You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Give a simple verbal description of the language recognized by the following DFA.

2. Draw NFAs for the following languages, taking full advantage of nondeterminism:
   a. strings over alphabet \{0, 1, \ldots, 9\} where the final digit has not appeared before;
   b. binary strings that begin or end with 00 or 11.
Prove that the following languages over the binary alphabet are regular:

(2 pts)  
(a) strings in which every 1 is immediately followed by 00;

(2 pts)  
(b) strings that contain both 010 and 101 as substrings;

(2 pts)  
(c) strings that end with 00.
4 Let $L_1$ and $L_2$ be languages that can be recognized by NFAs with $n_1$ and $n_2$ states, respectively. Prove that $L_1 \cap L_2$ can be recognized by a DFA with at most $2^{n_1 + n_2}$ states.

5 Prove that every language $L \subseteq \{0, 1\}^*$ can be expressed as a countable union $L = \bigcup_{i=1}^{\infty} R_i$ for some regular languages $R_1, R_2, R_3, \ldots$. 
6 For a language $L \subseteq \Sigma^*$, define $\text{delete}(L) = \{uv : u\sigma v \in L \text{ for some } \sigma \in \Sigma\}$. Thus, $\text{delete}(L)$ is the set of all strings obtained by taking a nonempty string in $L$ and deleting a single character from it. Prove that if $L$ is regular, so is $\text{delete}(L)$.

7 For a language $L$, define $\text{suffix}(L) = \{v : uv \in L \text{ for some } u\}$. Thus, $\text{suffix}(L)$ is the set of all suffixes of strings in $L$. Show how to transform a DFA for any given regular language $L$ into an NFA for $\text{suffix}(L)$.
SOLUTIONS
You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Give a simple verbal description of the language recognized by the following DFA.

\[ \begin{array}{c}
\text{1} & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array} \]

**Solution.** All binary strings that end with 0010.

2. Draw NFAs for the following languages, taking full advantage of nondeterminism:

   a. strings over alphabet \{0, 1, \ldots, 9\} where the final digit has *not* appeared before;
   b. binary strings that begin or end with 00 or 11.

**Solution.**

\[ \text{a.} \quad \text{b.} \]
Prove that the following languages over the binary alphabet are regular:

(2 pts) a. strings in which every 1 is immediately followed by 00;
(2 pts) b. strings that contain both 010 and 101 as substrings;
(2 pts) c. strings that end with 00.

**Solution.**

a. The language is recognized by the following NFA:

![NFA](image)

b. The language of strings containing 010 is recognized by

![NFA](image)

Analogously, the language of strings containing 101 is recognized by

![NFA](image)

Since regular languages are closed under intersection, the language in the problem statement is regular.

c. The language is recognized by the following NFA:

![NFA](image)
Let \( L_1 \) and \( L_2 \) be languages that can be recognized by NFAs with \( n_1 \) and \( n_2 \) states, respectively. Prove that \( L_1 \cap L_2 \) can be recognized by a DFA with at most \( 2^{n_1+n_2} \) states.

**Solution.** Using the method discussed in class, transform the NFAs for \( L_1 \) and \( L_2 \) to DFAs with at most \( 2^{n_1} \) and \( 2^{n_2} \) states, respectively. Then combine these DFAs to obtain a DFA for \( L_1 \cap L_2 \) with at most \( 2^{n_1} \times 2^{n_2} = 2^{n_1+n_2} \) states, using the Cartesian-product construction from class.

Prove that every language \( L \subseteq \{0, 1\}^\ast \) can be expressed as a countable union \( L = \bigcup_{i=1}^{\infty} R_i \) for some regular languages \( R_1, R_2, R_3, \ldots \).

**Solution.** For any string \( w \in \{0, 1\}^\ast \), the language \( \{w\} \) is regular because it is recognized by the NFA

![Diagram](image)

Now define \( R_i \) to be the language whose only string is the \( i \)th string in \( L \), in lexicographic order (with the understanding that \( R_i = \emptyset \) if \( L \) has fewer than \( i \) strings). Then

\[
L = \bigcup_{i=1}^{\infty} R_i,
\]

where each \( R_i \) is regular by above.
For a language $L \subseteq \Sigma^*$, define $\text{delete}(L) = \{uv : u\sigma v \in L \text{ for some } \sigma \in \Sigma\}$. Thus, $\text{delete}(L)$ is the set of all strings obtained by taking a nonempty string in $L$ and deleting a single character from it. Prove that if $L$ is regular, so is $\text{delete}(L)$.

**Solution:**

For a language $L$, define $\text{suffix}(L) = \{v : uv \in L \text{ for some } u\}$. Thus, $\text{suffix}(L)$ is the set of all suffixes of strings in $L$. Show how to transform a DFA for any given regular language $L$ into an NFA for $\text{suffix}(L)$.

**Solution.** Add $\varepsilon$-transitions from the initial state $q_0$ to every state reachable from $q_0$. 