You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1 Prove that the following languages are regular:

(2 pts) a. binary strings that contain neither 010101 nor 000111 as substrings;
(2 pts) b. nonempty binary strings in which the first and last symbols are different;
(2 pts) c. strings over the decimal alphabet \{0, 1, 2, \ldots, 9\} with symbols in sorted order.
2. Give a simple verbal description of the language recognized by the following DFA.

![DFA Diagram]

3. Draw NFAs for the following languages, taking full advantage of nondeterminism:

a. binary strings that contain a pair of 0s separated by two or more symbols;

b. even-length binary strings in which symbols come in pairs, e.g., \(\varepsilon, 00, 0011, 1111\), but not \(0, 01, 001101\).
Let $L$ be a regular language over a given alphabet $\Sigma$. Define $L'$ to be the set of all strings in $L$ whose length is not a multiple of 2015. Prove that $L'$ is regular.

Consider the language $L$ whose strings are binary encodings (with leading zeroes ignored) of multiples of 17. Thus, $L$ contains $\varepsilon, 0, 00, 10001, 00010001, 110011$ but not 10, 11, 0001. Prove that $L$ is regular.
For a language $L$ over the binary alphabet, let edit($L$) denote the set of strings that can be obtained from a string in $L$ by flipping exactly one bit. Prove that edit($L$) is regular whenever $L$ is regular.

Let $L$ be the language of strings over alphabet $\{a, b, c\}$ in which the number of $a$’s, the number of $b$’s, and the number of $c$’s are either all even, or precisely one of them is even. Give a DFA for $L$ with at most four states.
SOLUTIONS
You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Prove that the following languages are regular:

   (2 pts)  a. binary strings that contain neither 010101 nor 000111 as substrings;
   (2 pts)  b. nonempty binary strings in which the first and last symbols are different;
   (2 pts)  c. strings over the decimal alphabet {0, 1, 2, ..., 9} with symbols in sorted order.

Solution. The languages in (a)–(c) are given by

\[
\begin{align*}
\{0, 1\}* &\{010101\}\{0, 1\}* \cup \{0, 1\}*\{000111\}\{0, 1\}*, \\
\{0\}\{0, 1\}* &\{1\} \cup \{1\}\{0, 1\}*\{0\}, \\
\{0\}* &\{1\} \cdots \{9\}*,
\end{align*}
\]

respectively. Thus, they are obtained by applying the operations of Kleene star, concatenation, union, and complement to the regular languages \{0\}, \{1\}, ..., \{9\}, \{0, 1\}, \{010101\}, \{000111\}. Since regular languages are closed under these operations, the resulting languages are regular as well.
(3 pts) 2 Give a simple verbal description of the language recognized by the following DFA.

![DFA Diagram]

Solution. Binary strings in which the number of 0s is a multiple of 5.

3 Draw NFAs for the following languages, taking full advantage of nondeterminism:

(2 pts) a. binary strings that contain a pair of 0s separated by two or more symbols;
(2 pts) b. even-length binary strings in which symbols come in pairs, e.g., \( \varepsilon, 00, 0011, 1111 \), but not 0, 01, 001101.

![NFA Diagram a]

a.

![NFA Diagram b]

b.
Let $L$ be a regular language over a given alphabet $\Sigma$. Define $L'$ to be the set of all strings in $L$ whose length is not a multiple of 2015. Prove that $L'$ is regular.

**Solution.** The language, call it $A$, of strings whose length is not a multiple of 2015 is regular, with DFA $(\{0, 1, 2, \ldots, 2014\}, \Sigma, 0, \{1, 2, \ldots, 2014\})$ where $\delta(q, \sigma) = (q + 1) \mod 2015$. Since $L' = L \cap A$ and regular languages are closed under intersection, it follows that $L'$ is regular.

Consider the language $L$ whose strings are binary encodings (with leading zeroes ignored) of multiples of 17. Thus, $L$ contains $\varepsilon, 0, 00, 10001, 000010001, 110011$ but not $10, 11, 0001$. Prove that $L$ is regular.

**Solution.** We will give a DFA for $L$. The idea is to convert the input string on the fly to a decimal number and check to see if that number is a multiple of 17. A direct implementation of this idea results in the “infinite” automaton $(\{0, 1, 2, \ldots\}, \{0, 1\}, \delta, 0, \{0, 17, 34, \ldots\})$, where $\delta(q, \sigma) = 2q + \sigma$. The only flaw in this construction is that it uses infinitely many states.

To get an actual DFA for $L$, recall that we only care about the remainder of the decimal number modulo 17. Therefore, we can perform all intermediate computations modulo 17. This corresponds to the automaton $(\{0, 1, 2, \ldots, 16\}, \{0, 1\}, \delta, 0, \{0\})$, where $\delta(q, \sigma) = (2q + \sigma) \mod 17.$
6. For a language $L$ over the binary alphabet, let $\text{edit}(L)$ denote the set of strings that can be obtained from a string in $L$ by flipping exactly one bit. Prove that $\text{edit}(L)$ is regular whenever $L$ is regular.

**Solution.** A DFA for $L$ can be transformed into an NFA for $\text{edit}(L)$ as follows, where $\delta$ refers to the DFA’s transition function.

Let $L$ be the language of strings over alphabet $\{a, b, c\}$ in which the number of $a$’s, the number of $b$’s, and the number of $c$’s are either all even, or precisely one of them is even. Give a DFA for $L$ with at most four states.

**Solution.**