You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1. Give a regular expression for each of the following languages:

   (2 pts) a. strings of length at most 2015 over a given alphabet $\Sigma$;
   (3 pts) b. binary strings in which the number of 0s and the number of 1s are not both odd.
(2 pts)  2  Simplify \((0 \cup 1)^*01(0 \cup 1)^* \cup 1^*0^*\) as much as possible, and explain your answer in detail.

(3 pts)  3  Prove or disprove: the Kleene star of every language is a regular language.
(3 pts) Let $L$ be the language of palindromes over $\{0, 1\}$. Determine the equivalence classes of $\equiv_L$.

(4 pts) Construct a DFA for $0^*1^*0^+$ with the smallest possible number of states. Prove that your DFA is the smallest possible.
For each of the following languages, determine whether it is regular, and prove your answer:

(2 pts) a. binary strings in which the number of 1s is a multiple of the number of 0s;
(2 pts) b. binary strings of the form $uvuw$, where $u, v, w$ are strings and $u$ is nonempty;
(2 pts) c. nonempty binary strings of even length with the two middle symbols unequal;
(2 pts) d. strings over the alphabet $\{a, b, c\}$ that contain each of the three alphabet symbols.
SOLUTIONS
You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1 Give a regular expression for each of the following languages:

(2 pts) a. strings of length at most 2015 over a given alphabet \( \Sigma \);

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Solution.

a. \((\Sigma \cup \varepsilon)^{2015}\)

b. \(1^*(01^*01^*)^* \cup 0^*(10^*10^*)^*\). There is a hard way to solve this problem and an easy one. The hard way is to construct an NFA and convert it to a regular expression. The easy way is to observe that this language is the union of two simpler languages: (i) binary strings with an even number of 0s, and (ii) binary strings with an even number of 1s.
Simplify \((0 \cup 1)*01(0 \cup 1)^* \cup 1^*0^*\) as much as possible, and explain your answer in detail.

**Solution.** The regular expression \((0 \cup 1)*01(0 \cup 1)^*\) corresponds to binary strings that contain 01. The only strings that do not have this property are of the form 1*0*. Thus, the union of \((0 \cup 1)*01(0 \cup 1)^*\) and 1*0* simplifies to \((0 \cup 1)^*\).

Prove or disprove: the Kleene star of every language is a regular language.

**Solution.** The claim is false. Consider \(L = \{0^n1^n : n \geq 0\}\). Then \(L^*\) is the set of strings of the form \(0^{n_1}1^{n_1}0^{n_2}1^{n_2} \ldots 0^{n_k}1^{n_k}\) for some \(n_1, n_2, \ldots, n_k\). Let \(p\) be arbitrary and consider the string \(w = 0^p1^p \in L^*\). If \(x, y, z\) are strings such that \(y\) is nonempty, \(|xy| \leq p\), and \(w = xyz\), then \(xy^2z \notin L^*\). Therefore by the pumping lemma, \(L^*\) is nonregular.
4 Let $L$ be the language of palindromes over \{0, 1\}. Determine the equivalence classes of $\equiv_L$.

Solution. Let $u, v$ be arbitrary strings with $u \neq v$. Then for $N$ sufficiently large,

$$u 01^N 0 u^R \in L,$$

$$v 01^N 0 u^R \notin L.$$

Thus, every string is in an equivalence class by itself.

Note. A common mistake is to claim that $uu^R \in L, u^R \notin L$ for any pair of distinct strings $u, v$. This claim fails for many string pairs, e.g., 0, 00 as well as 0, 010101.

5 Construct a DFA for $0^*1^*0^+$ with the smallest possible number of states. Prove that your DFA is the smallest possible.

Solution. The following DFA with five states recognizes $0^*1^*0^+$:

```
0 1 0 1 0,1
0 - 1 - 0 - 1 -
0 - 1 - 0 - 1 -
0 - 1 - 0 - 1 -
0 - 1 - 0 - 1 -
```

The five strings $\varepsilon, 0, 1, 10, 101$ are in pairwise distinct equivalence classes of $\equiv_{0^*1^*0^+}$, with distinguishing suffixes given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon$</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>101</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$\varepsilon$</td>
<td>010</td>
<td>010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>010</td>
<td>010</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>101</td>
<td>00</td>
<td>00</td>
<td>00</td>
<td>00</td>
<td>00</td>
</tr>
</tbody>
</table>

By the Myhill-Nerode theorem, we conclude that no smaller DFA exists.
For each of the following languages, determine whether it is regular, and prove your answer:

(2 pts) a. binary strings in which the number of 1s is a multiple of the number of 0s;
(2 pts) b. binary strings of the form $uvw$, where $u$, $v$, $w$ are strings and $u$ is nonempty;
(2 pts) c. nonempty binary strings of even length with the two middle symbols unequal;
(2 pts) d. strings over the alphabet $\{a, b, c\}$ that contain each of the three alphabet symbols.

Solution.

In each part, $L$ stands for the language in question.

a. Nonregular. For any positive integers $i < j$, we have $0^i1^i \in L$ but $0^j1^i \notin L$. Therefore, the strings $0, 00, 000, \ldots, 0^n, \ldots$ are each in a distinct equivalence class of $\equiv_L$. By the Myhill-Nerode theorem, $L$ is nonregular.

b. Regular. This language contains precisely those strings in which the first symbol occurs again, which corresponds to the regular expression $1^*1\Sigma^* \cup 0\Sigma^*0\Sigma^*$.

c. Nonregular. For any even positive integers $i \neq j$, we have $0^i1^i \in L$ but $0^j1^i \notin L$. Therefore, the strings $0^2, 0^4, \ldots, 0^{2n}, \ldots$ are each in a distinct equivalence class of $\equiv_L$. By the Myhill-Nerode theorem, $L$ is nonregular.

d. Regular. The language is given by $\{a, b\}^* \cup \{a, c\}^* \cup \{b, c\}^*$. Thus, it is obtained by applying the operations of Kleene star, union, and complement to the regular languages $\{a, b\}, \{a, c\}, \{b, c\}$. Since regular languages are closed under these operations, the result is regular.