You have 90 minutes to complete this exam. You may state without proof any fact taught in class or assigned as homework.

(3 pts) 1 Give a simple verbal description of the language recognized by the following NFA:

![NFA Diagram]

*Solution:* binary strings that do not start with 11111.

2 Draw NFAs for the following languages, taking full advantage of nondeterminism:

(2 pts) a. binary strings that have odd length or contain at most one 1;

(2 pts) b. binary strings in which every 0 is immediately preceded and immediately followed by a 1.

*Solution.*

a. 

![NFA Diagram]

b. 

![NFA Diagram]
Prove that the following languages $L$ are regular:

(2 pts) a. binary strings that begin and end with 010;
(2 pts) b. binary strings in which 01 occurs more times than 10;
(2 pts) c. decimal strings in which every digit (0, 1, 2, ..., 9) occurs at most once.

**Solution.**

**a.** The language of strings that begin with 010 is regular, with NFA

![NFA](image)

The language of strings that end with 010 is also regular, with NFA

![NFA](image)

The intersection of these two languages is the language $L$ in the problem statement. This makes $L$ regular because regular languages are closed under intersection.

**b.** This language is recognized by the following NFA:

![NFA](image)

**c.** Any decimal string of length greater than 10 will contain a pair of identical symbols. It follows that the strings in $L$ are bounded in length by 10, which makes $L$ finite. We proved in class that every finite language is regular.
For a binary string $w$, its \textit{bitwise complement} is denoted $\overline{w}$ and defined as the string obtained by flipping every bit of $w$. Prove that for every regular language $L$ over the binary alphabet, the language $L' = \{\overline{w} : w \in L\}$ is also regular.

**Solution.** Starting with a DFA for $L$, swap the edge labels 0 and 1 out of every state. The resulting DFA recognizes $L'$. Formally, let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA for $L$. Then $L'$ is recognized by $(Q, \Sigma, \delta', q_0, F)$, where $\delta'(q, \sigma) = \delta(q, \overline{\sigma})$.

Let $L$ be a regular language. Define $L^\dagger$ to be the set of all strings that can be obtained by concatenating one or more nonempty strings in $L$. Prove that $L^\dagger$ is regular.

**Solution.** Note that $L^\dagger = L^* \setminus \{\varepsilon\}$, where $L$ and $\{\varepsilon\}$ are regular. Since regular languages are closed under Kleene star and set difference, $L^\dagger$ is also regular.

**An incorrect solution.** Starting with a DFA for $L$, make the initial state rejecting and add $\varepsilon$-transitions from every accepting state back to the initial state. It is tempting to claim that the resulting automaton recognizes $L^\dagger$. This is incorrect in general. The new automaton may not accept all strings in $L^\dagger$. 

For a language \( L \), define \( \text{core}(L) \) as the set of all strings \( v \) such that \( uwv \in L \) for some nonempty strings \( u \) and \( w \). Prove that \( \text{core}(L) \) is regular for every regular language \( L \).

**Solution.** Let \((Q, \Sigma, \delta, q_0, F)\) be a DFA for \( L \). To recognize \( \text{core}(L) \), we make two changes to this DFA. First, we add a new start state \( q_{\text{new}} \) and link it with \( \varepsilon \)-transitions to every state of \( Q \) that is reachable from \( q_0 \) via a nonempty path. Second, we redefine the accept/reject states in this augmented automaton, marking as accepting those states of \( Q \) that have a nonempty path to a state of \( F \).

Fix an arbitrary NFA \( N \). We say that \( N \) strongly accepts a given string \( w \) iff \( N \) can end up in two or more accept states after processing \( w \). Show that \( N \)'s strongly accepted strings form a regular language.

**Solution.** Let \( N = (Q, \Sigma, \delta, q_0, F) \) be the given NFA. Use the construction from class to convert \( N \) to an equivalent DFA: \( D = (\mathcal{P}(Q), \Sigma, \Delta, S_0, \mathcal{F}) \). Then \( N \) strongly accepts a string \( w \) iff \( D \)'s end state on \( w \) contains two or more of the NFA's accept states, \( F \). Therefore, \( N \)'s strongly accepted strings are recognized by the DFA \((\mathcal{P}(Q), \Sigma, \Delta, S_0, \mathcal{F}')\), where \( \mathcal{F}' = \{ S \subseteq Q : |S \cap F| \geq 2 \} \).