1. Give a regular expression for each of the following languages:

   (2 pts) a. binary strings other than 01;
   (2 pts) b. binary strings that do not contain 100 as a substring;
   (2 pts) c. strings over the alphabet \{a, b, c\} that contain all three alphabet symbols.

**Solution.**

   a. \(\varepsilon \cup \Sigma \cup 1\Sigma \cup \Sigma 0 \cup \Sigma^3\Sigma^*\)
   b. \(0^*(1 \cup 10)^*\)
   c. \(a\Sigma^*b\Sigma^*c\Sigma^* \cup a\Sigma^*c\Sigma^*b\Sigma^* \cup b\Sigma^*a\Sigma^*c\Sigma^* \cup b\Sigma^*c\Sigma^*a\Sigma^* \cup c\Sigma^*a\Sigma^*b\Sigma^* \cup c\Sigma^*b\Sigma^*a\Sigma^*\)

(3 pts) 2. Let \(L\) be the language of strings over the alphabet \{0, 1, 2, \ldots, 9\} in which no two adjacent symbols are the same. Determine the equivalence classes of \(\equiv_L\).

**Solution.** Decimal strings that contain two identical symbols next to each other are all \(L\)-indistinguishable. Thus, the strings in \(\overline{L}\) form a single class. Of the remaining strings, a pair \(u\) and \(v\) are \(L\)-distinguishable iff they disagree in their last digit (if \(u\) ends with digit \(i\), and \(v\) does not, then \(ui \notin L\) and \(vi \in L\)). Summarizing, the classes are \(\overline{L}, L_0, L_1, L_2, \ldots, L_9, \{\varepsilon\}\), where \(L_i\) denotes the strings in \(L\) that end with \(i\).
3. Construct the smallest possible DFA for the language of odd-length binary strings that start with 01. Prove that your DFA is indeed the smallest possible.

**Solution.** The language \( L \) in question is recognized by the following DFA:

![DFA Diagram]

By the Myhill–Nerode theorem, no smaller DFA exists for \( L \) because each of the five strings \( \varepsilon, 0, 1, 01, 010 \) is in a different equivalence class of \( \equiv_L \). The distinguishing suffixes are as follows:

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<th>( \varepsilon )</th>
<th>0</th>
<th>1</th>
<th>01</th>
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</table>

4. Let \( L \) be the language of binary strings in which the number of 0s is positive and divides the number of 1s. Use the pumping lemma to prove that \( L \) is nonregular. You must not use the Myhill–Nerode theorem or closure properties.

**Solution.** Let \( p \geq 1 \) be arbitrary. Consider the string \( w = 0^p1^p \in L \). Let \( w = xyz \) be any decomposition such that \( y \) is nonempty and occurs within the first \( p \) zeroes. Then pumping up results in a string with more 0s than 1s, ensuring that the number of 0s no longer divides the number of 1s. Thus, \( xy^2z \notin L \). By the pumping lemma, \( L \) is nonregular. (Note that pumping down would not work here!)
5 Prove or disprove: for any nonregular language $L$, the palindromes in $L$ also form a nonregular language.

Solution. False. Let $L = \{0^n1^n : n \geq 0\}$. We proved in class that $L$ is nonregular. Yet, the palindromes in $L$ form the regular language $\{\varepsilon\}$.

6 For a language $L$, let $\text{permute}(L)$ denote the set of all strings that can be obtained by taking a string in $L$ and keeping it as is or reordering its symbols. For example, $\text{permute}(\{\varepsilon, ab, abb\}) = \{\varepsilon, ab, ba, abb, bab, bba\}$. Prove that regular languages are not closed under the permute operation.

Solution. Consider the regular language $L = (01)^*$. Then $\text{permute}(L)$ is the language of binary strings with as many 0s as 1s, which we proved in class is nonregular.
For each of the following languages $L$ over the binary alphabet, determine whether it is regular and prove your answer:

(2 pts) a. strings of the form $wwu$, where $w \in \{0,1\}^+$ and $u \in \{0,1\}^*$;

(2 pts) b. strings that are palindromes or contain 00 as a substring;

(2 pts) c. strings that contain a nonempty substring with equally many 0s and 1s.

Solution.

a. Nonregular. For any positive integers $i < j$, we have $01^i 01^i \in L$ but $01^j 01^i \notin L$. Therefore, each of the strings $01, 011, 0111, \ldots$ is in a different equivalence class of $\equiv_L$. Since there are infinitely many equivalence classes, $L$ is nonregular by the Myhill–Nerode theorem.

b. Nonregular. For any nonnegative integers $i \neq j$, we have $1^i 01^i \in L$ but $1^j 01^i \notin L$. Therefore, each of the strings $\varepsilon, 1, 11, 111, 1111, \ldots$ is in a different equivalence class of $\equiv_L$. Since there are infinitely many equivalence classes, $L$ is nonregular by the Myhill–Nerode theorem.

c. Regular, with regular expression $\Sigma^* (01 \cup 10) \Sigma^*$. Indeed, $L$ by definition contains every string that has 01 or 10 as a substring. Conversely, any string $w \in L$ must contain both 0s and 1s, which means that $w$ must contain 01 or 10.