CS6220: DATA MINING TECHNIQUES

Matrix Data: Clustering: Part 2

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Reminder

- Course project proposal due this Wednesday (10/16)
  - Submit pdf file in blackboard
  - Sign-up for discussions on next Friday (15mins for each group)
  - Meet in my office
- Homework 2 will be out today, and will be due two weeks later

Matrix Data: Clustering: Part 2

- Revisit K-means
- Mixture Model and EM algorithm
- Kernel K-means
- Summary

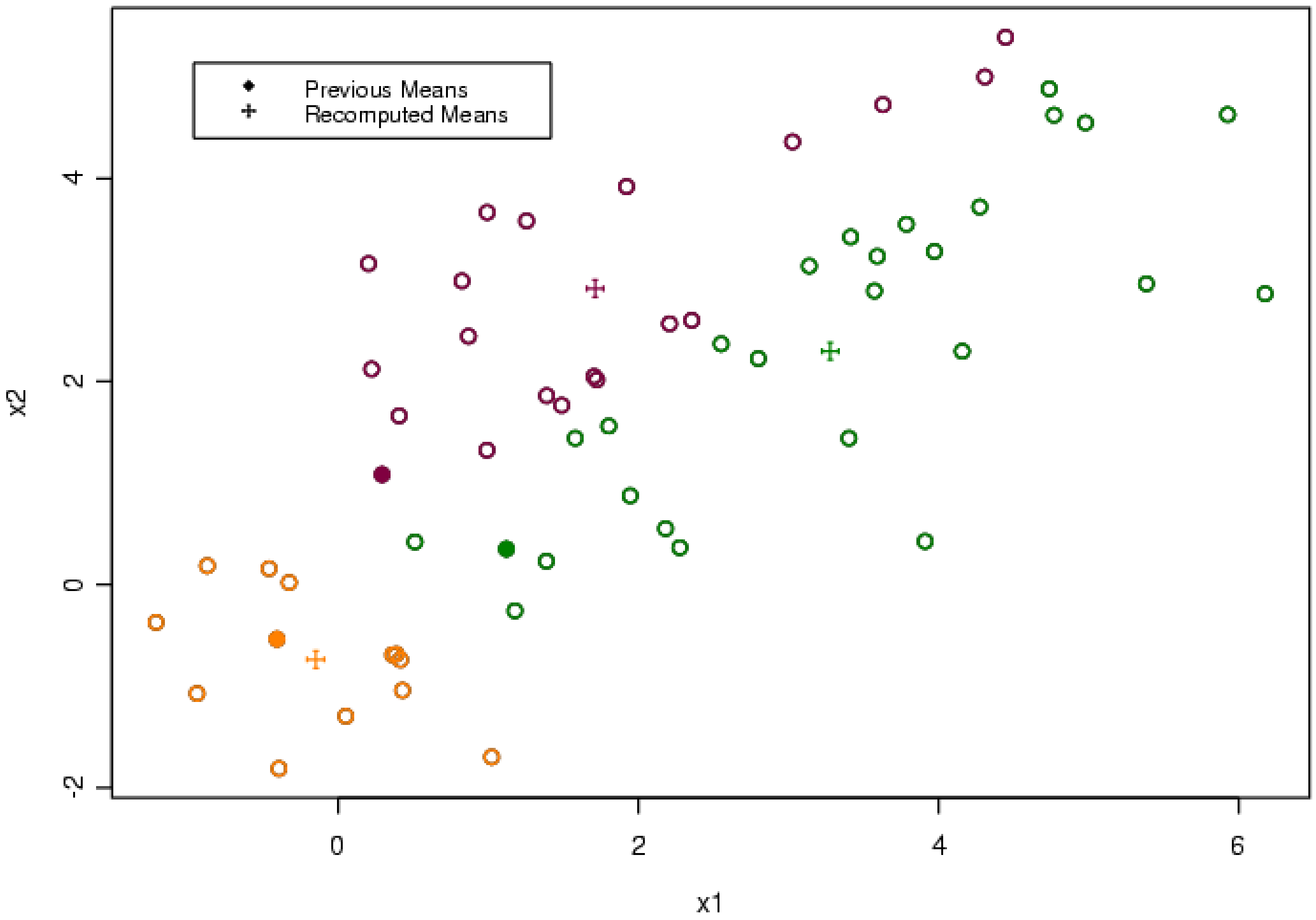
Recall K-Means

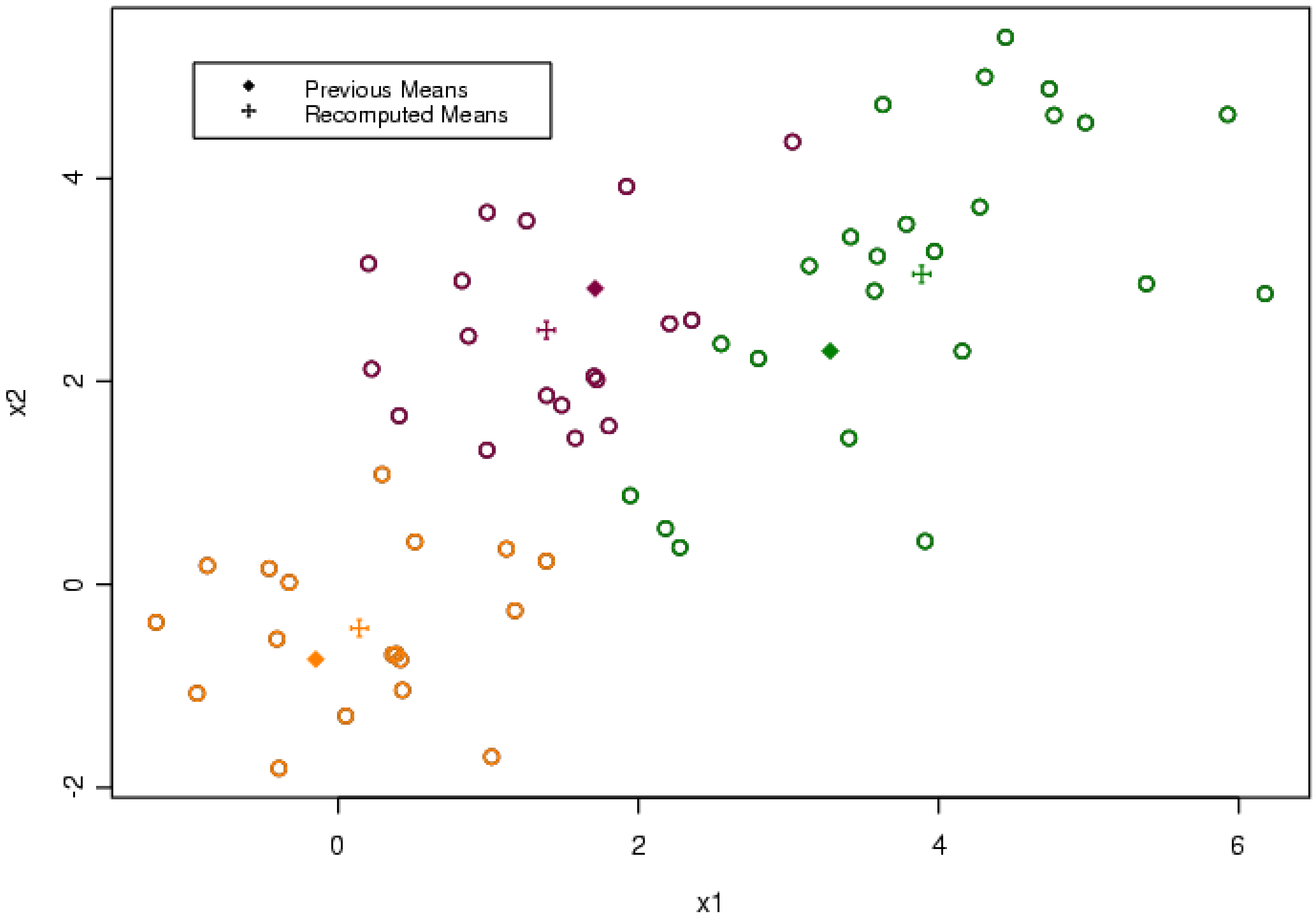
- Objective function
  - $J = \sum_{j=1}^{k} \sum_{C(i)=j} ||x_i - c_j||^2$
  - Total within-cluster variance
- Re-arrange the objective function
  - $J = \sum_{j=1}^{k} \sum_{i} w_{ij} ||x_i - c_j||^2$
    - $w_{ij} \in \{0,1\}$
    - $w_{ij} = 1, if\ x_i\ belongs\ to\ cluster\ j; w_{ij} = 0, otherwise$
  - Looking for:
    - The best assignment w_{ij}
    - The best center c_j

Solution of K-Means

$$J = \sum_{j=1}^{k} \sum_{i} w_{ij} ||x_i - c_j||^2$$

- Iterations
  - Step 1: Fix centers c_j, find assignment w_{ij} that minimizes J
    - => $w_{ij} = 1, if\ ||x_i - c_j||^2$ is the smallest
  - Step 2: Fix assignment w_{ij}, find centers that minimize J
    - => first derivative of J = 0
    - => $\frac{\partial J}{\partial c_j} = -2 \sum_i w_{ij}(x_i - c_j) = 0$
    - =>$c_j = \frac{\sum_i w_{ij} x_i}{\sum_i w_{ij}}$
      - Note $\sum_i w_{ij}$ is the total number of objects in cluster j


Previous Means
Recomputed Means
x1
x2
0
2
4
6
-2
0
2
4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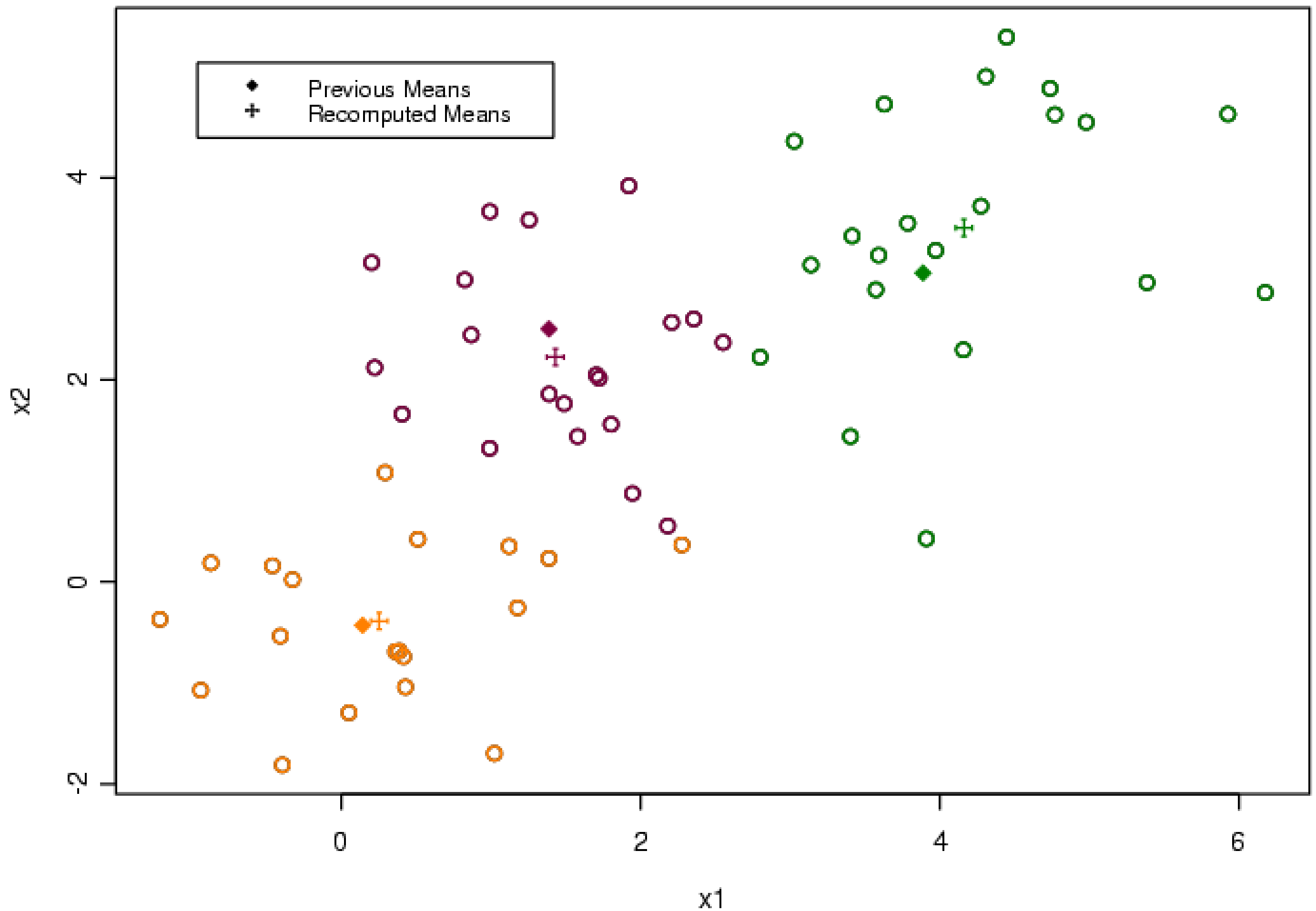
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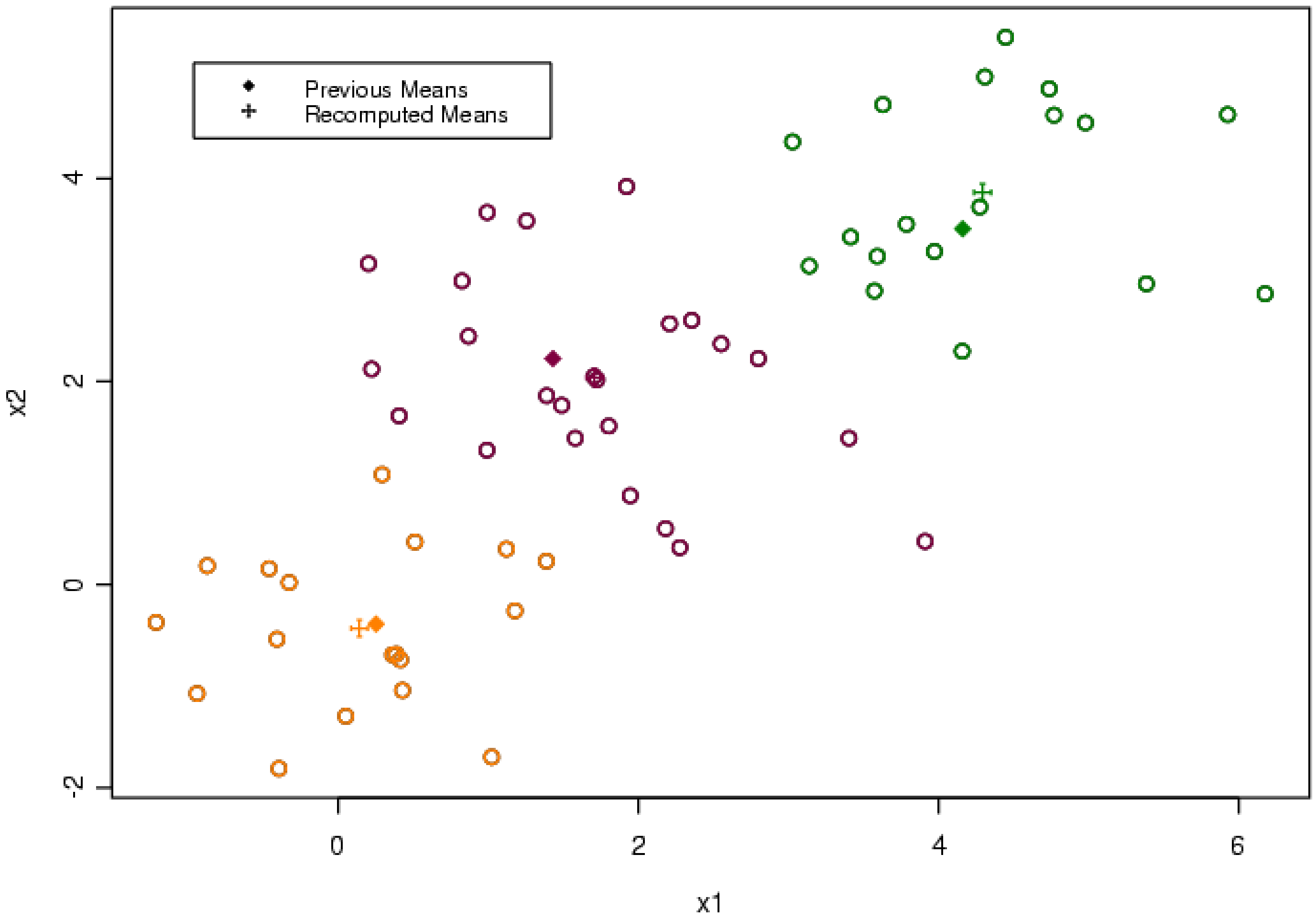
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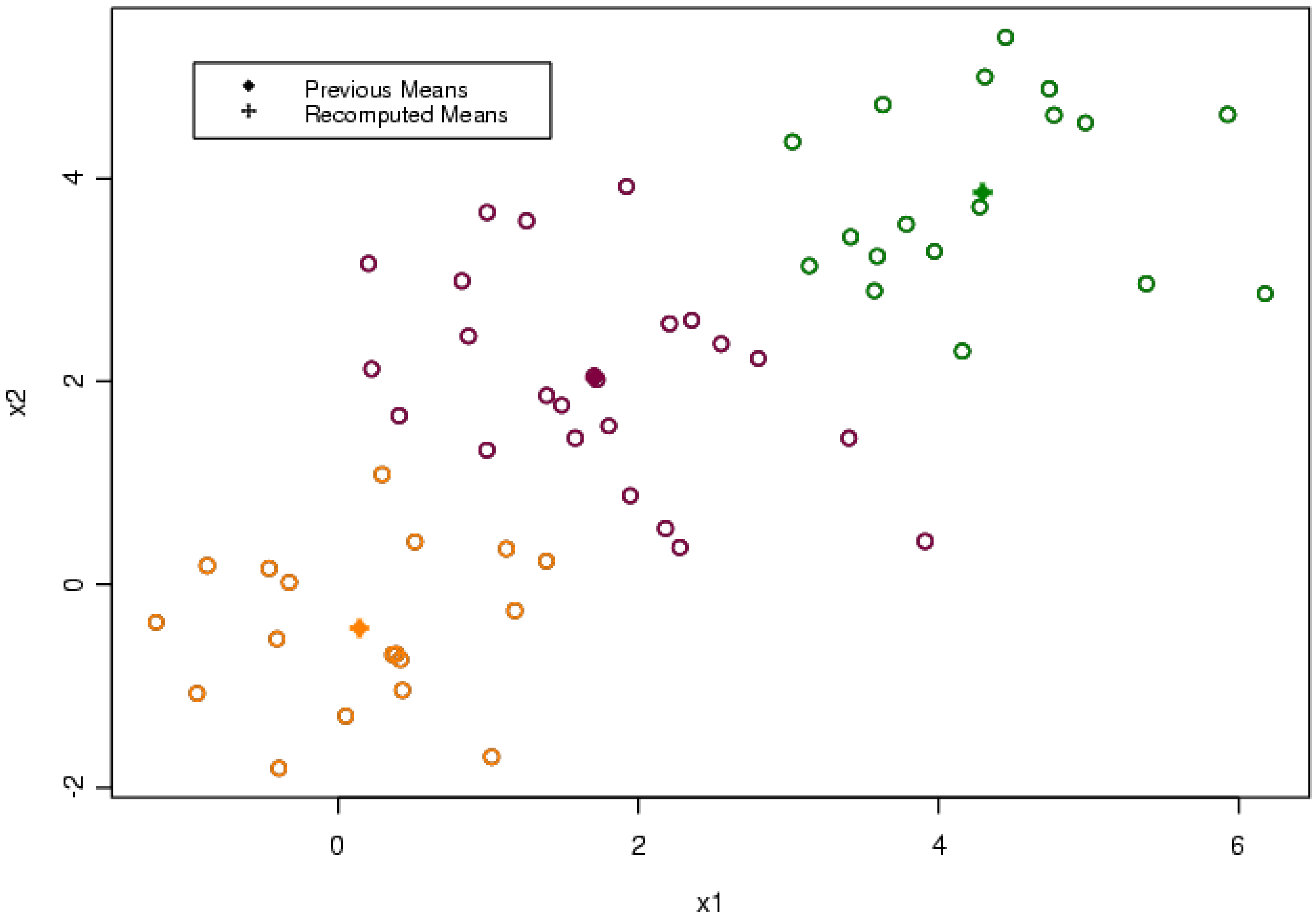

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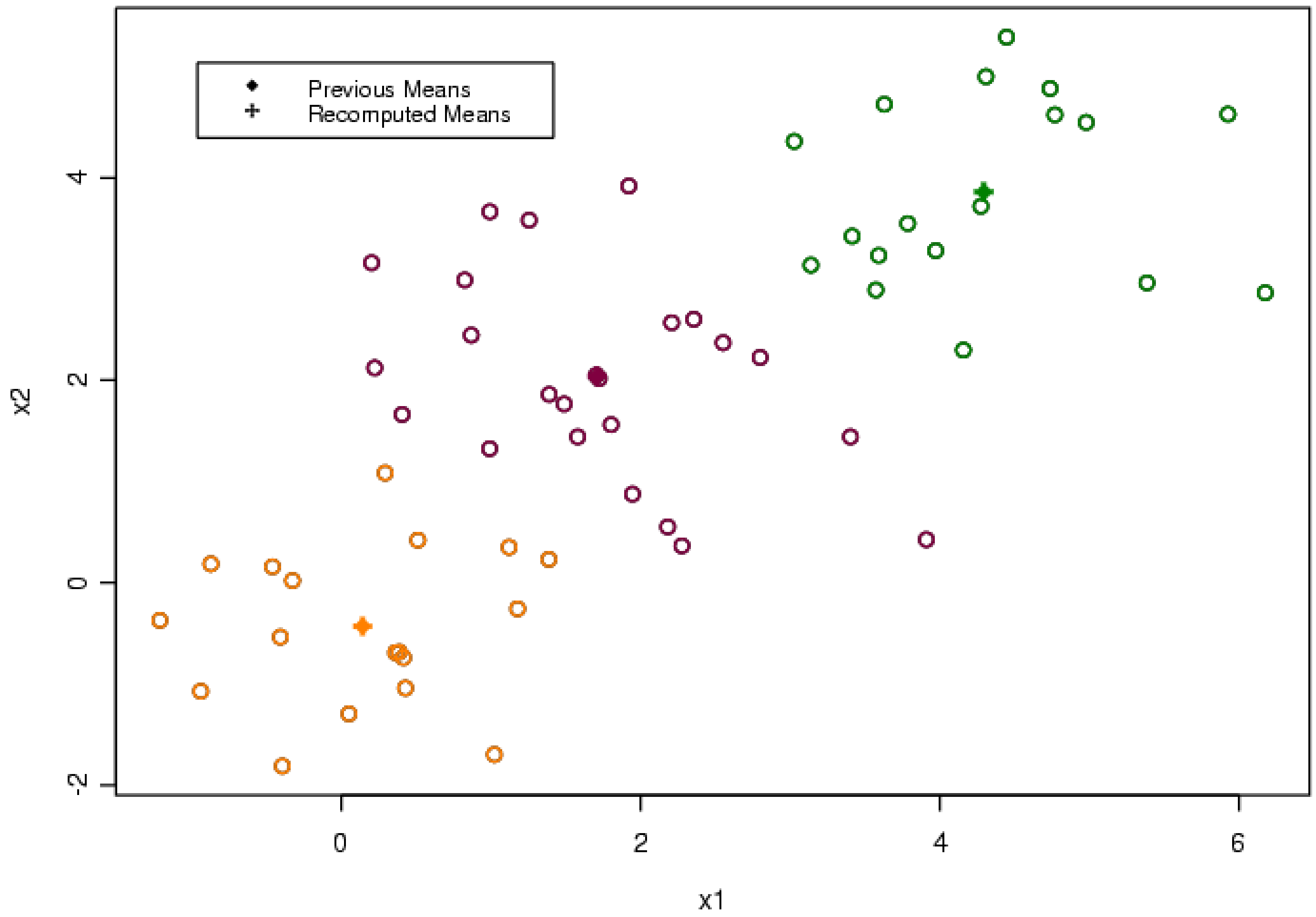
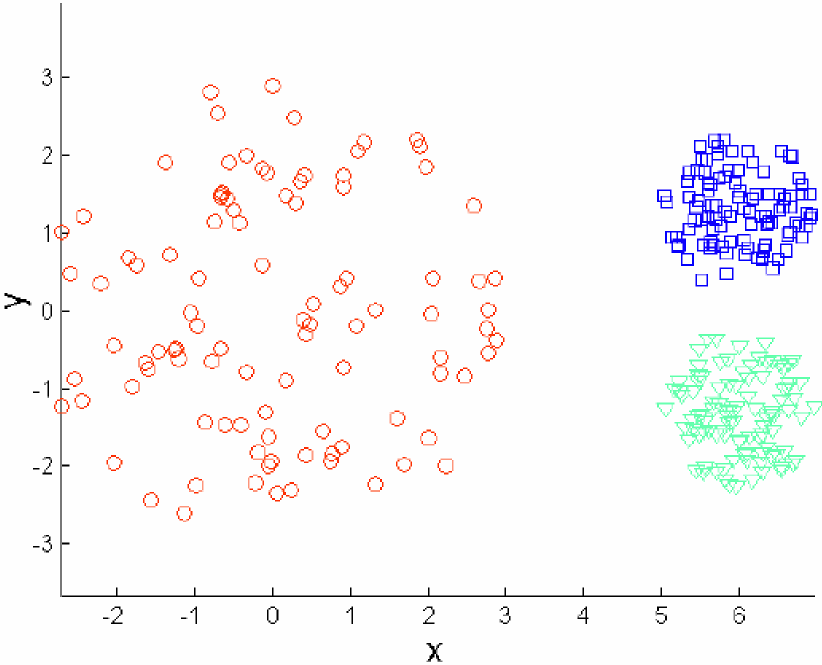


Converges! Why?

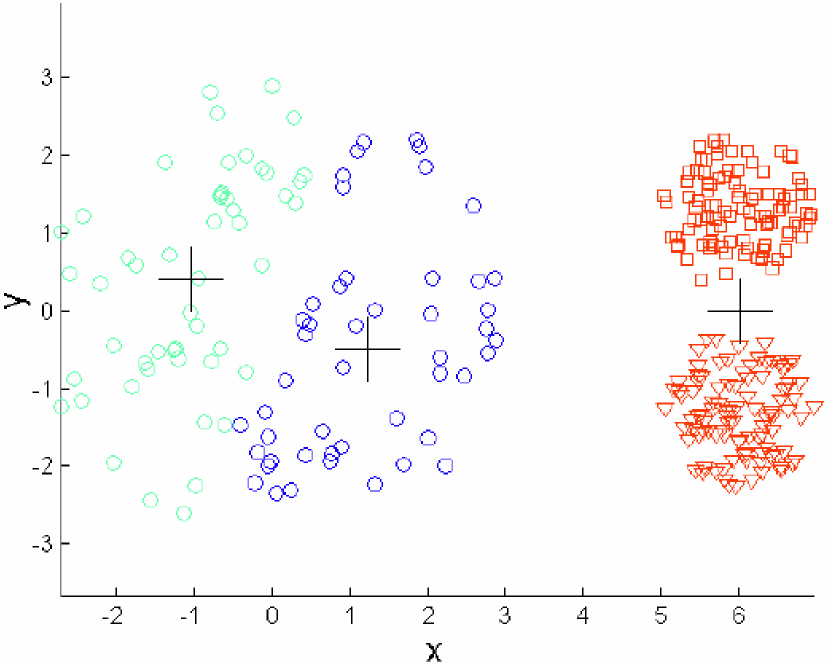
Limitations of K-Means

- K-means has problems when clusters are of differing
  - Sizes
  - Densities
  - Non-Spherical Shapes

Limitations of K-Means: Different Density and Size

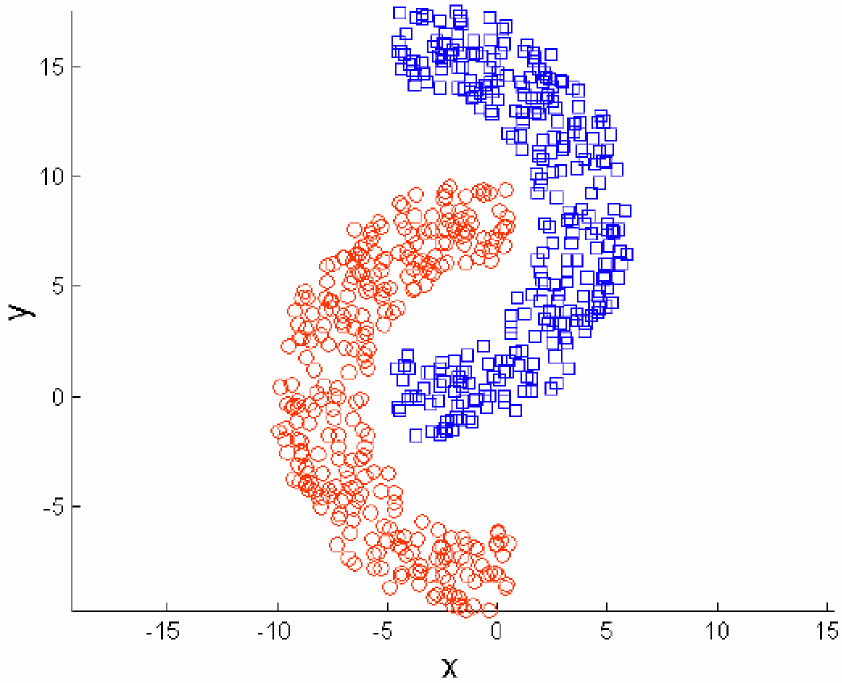


Original Points

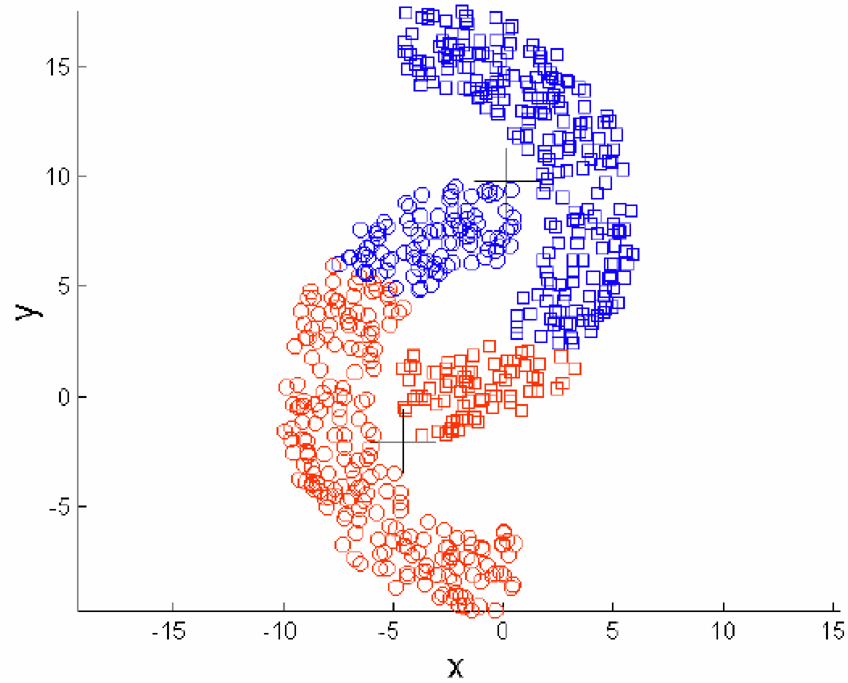


K-means (3 Clusters)

Limitations of K-Means: Non-Spherical Shapes



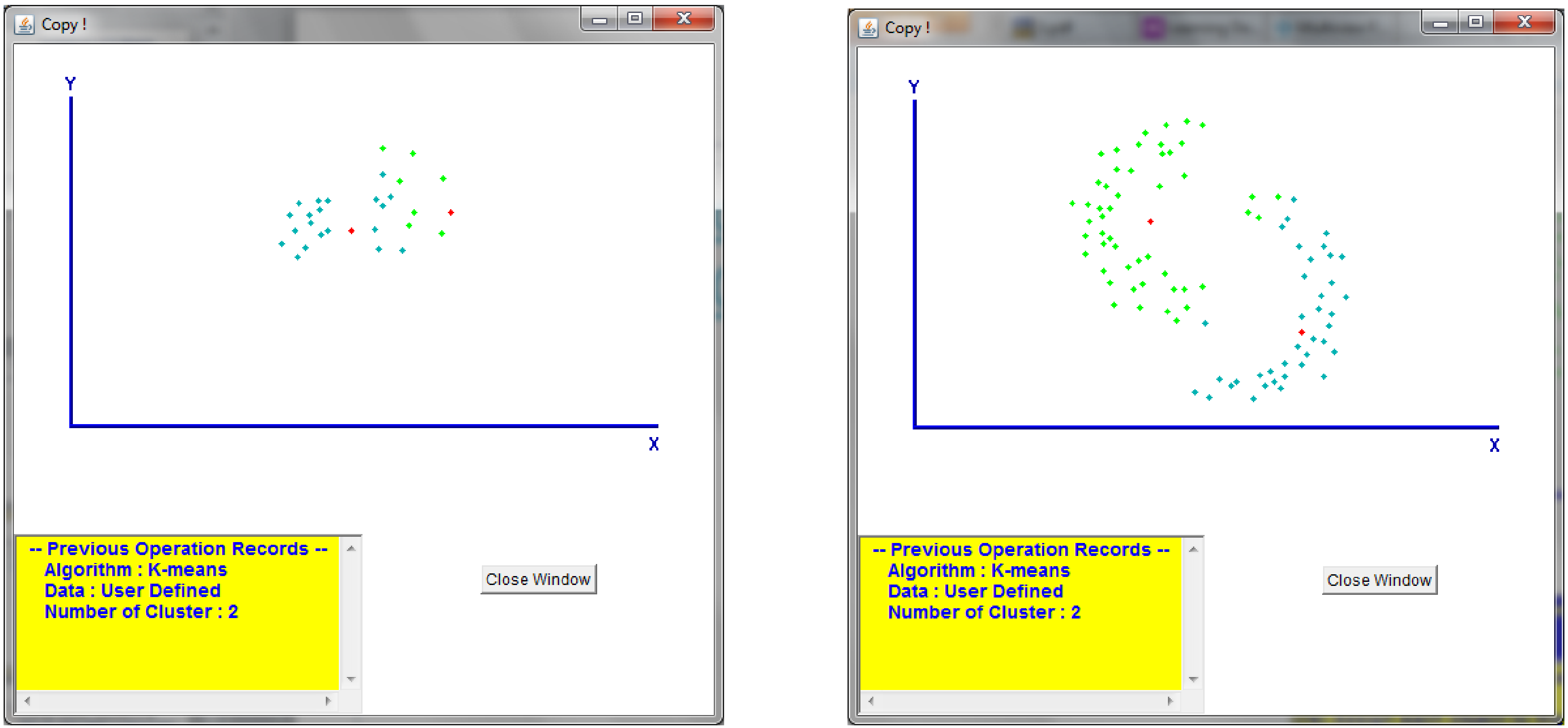
Original Points



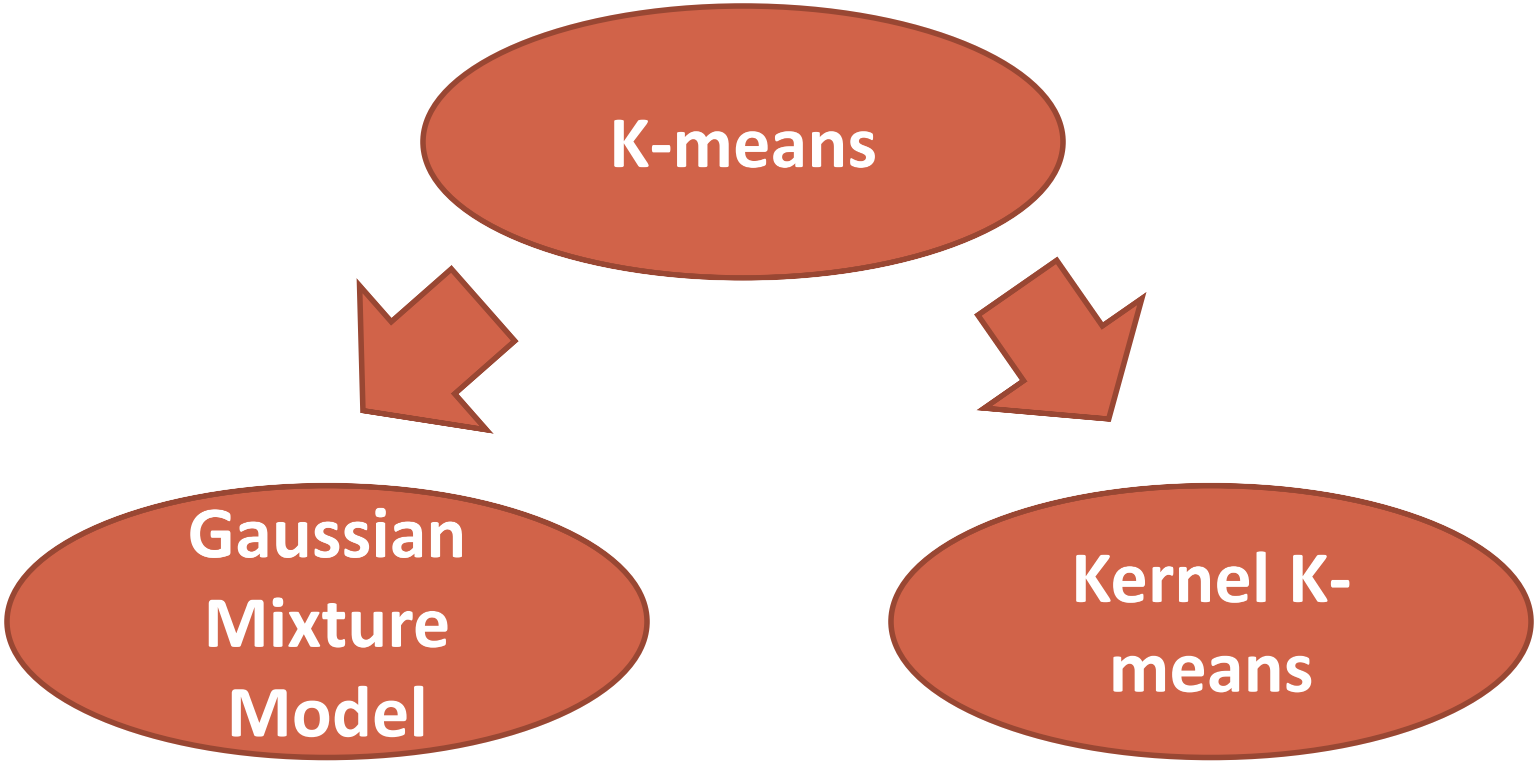
K-means (2 Clusters)

Demo

- http://webdocs.cs.ualberta.ca/~yaling/Cluster/Applet/Code/Cluster.html



Connections of K-means to Other Methods



Matrix Data: Clustering: Part 2

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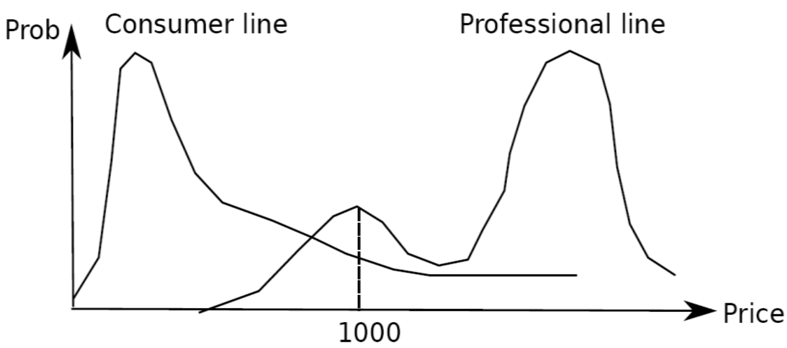
Fuzzy Set and Fuzzy Cluster

- Clustering methods discussed so far
  - Every data object is assigned to exactly one cluster
- Some applications may need for fuzzy or soft cluster assignment
  - Ex. An e-game could belong to both entertainment and software
- Methods: fuzzy clusters and probabilistic model-based clusters
- Fuzzy cluster: A fuzzy set S: $F_S : X \rightarrow [0, 1]$ (value between 0 and 1)

Probabilistic Model-Based Clustering

- Cluster analysis is to find hidden categories.
- A hidden category (i.e., *probabilistic cluster)* is a distribution over the data space, which can be mathematically represented using a probability density function (or distribution function).

- Ex. categories for digital cameras sold
  - consumer line vs. professional line
  - density functions f_1, f_2 for C_1, C_2
  - obtained by probabilistic clustering



- A **mixture model** assumes that a set of observed objects is a mixture of instances from multiple probabilistic clusters, and conceptually each observed object is generated independently
- **Our task**: infer a set of *k* probabilistic clusters that is mostly likely to generate *D* using the above data generation process

Mixture Model-Based Clustering

- A set C of k probabilistic clusters $C_1, \ldots, C_k$ with probability density functions $f_1, \ldots, f_k$, respectively, and their probabilities $\omega_1, \ldots, \omega_k$.
- Probability of an object o generated by cluster C_j is: $P(o|C_j) = \omega_j f_j(o)$
- Probability of o generated by the set of cluster $\boldsymbol{C}$ is: $P(o|\mathbf{C}) = \sum_{j=1}^{k} \omega_j f_j(o)$ $\sum_j w_j = 1$

Maximum Likelihood Estimation

- Since objects are assumed to be generated independently, for a data set D = {o_1, …, o_n}, we have,

$$P(D|\boldsymbol{C}) = \prod_{i=1}^{n} P(o_i|\boldsymbol{C}) = \prod_{i=1}^{n} \sum_{j=1}^{k} \omega_j f_j(o_i)$$

- Task: Find a set *C* of *k* probabilistic clusters s.t. *P*(*D*|***C***) is maximized

The EM (Expectation Maximization) Algorithm

- **The (EM) algorithm:** A framework to approach maximum likelihood or maximum a posteriori estimates of parameters in statistical models.
  - **E-step** assigns objects to clusters according to the current fuzzy clustering or parameters of probabilistic clusters
    - $w_{ij}^t = p\left(z_i = j \middle| \theta_j^t, x_i\right) \propto p\left(x_i \middle| C_j^t, \theta_j^t\right) p(C_j^t)$
  - **M-step** finds the new clustering or parameters that maximize the expected likelihood

Case 1: Gaussian Mixture Model

- Generative model
  - For each object:
    - Pick its distribution component: $Z \sim Multi(\omega_1, \ldots, \omega_k)$
    - Sample a value from the selected distribution: $X \sim N(\mu_Z, \sigma_Z^2)$
- Overall likelihood function
  - $L(D|\theta) = \prod_i \sum_j \omega_j p(x_i|\mu_j, \sigma_j^2)$
  - Q: What is θ here?

Estimating Parameters

- $l(D;\theta) = \sum_i \log \sum_j \omega_j p(x_i|\mu_j,\sigma_j^2)$ Intractable!
  - Considering the first derivative of μ_j:
  - $\frac{\partial l}{\partial u_j} = \sum_i \frac{\omega_j}{\sum_j \omega_j p(x_i|\mu_j,\sigma_j^2)} \frac{\partial p(x_i|\mu_j,\sigma_j^2)}{\partial \mu_j}$
  - $= \sum_i \frac{\omega_j p(x_i|\mu_j,\sigma_j^2)}{\sum_j \omega_j p(x_i|\mu_j,\sigma_j^2)} \frac{1}{p(x_i|\mu_j,\sigma_j^2)} \frac{\partial p(x_i|\mu_j,\sigma_j^2)}{\partial \mu_j}$
  - $= \sum_i \boxed{\frac{\omega_j p(x_i|\mu_j,\sigma_j^2)}{\sum_j \omega_j p(x_i|\mu_j,\sigma_j^2)}} \boxed{\frac{\partial log p(x_i|\mu_j,\sigma_j^2)}{\partial u_j}}$

$w_{ij} = P(Z=j|X=x_i,\theta)$

$\partial l(x_i)/\partial \mu_j$

Apply EM algorithm

- An iterative algorithm
  - E(expectation)-step
    - Evaluate the weight w_{ij} when $\mu_j, \sigma_j, \omega_j$ are given
    - $w_{ij} = \frac{\omega_j p(x_i|\mu_j, \sigma_j^2)}{\sum_j \omega_j p(x_i|\mu_j, \sigma_j^2)}$
  - M(maximization)-step
    - Evaluate $\mu_j, \sigma_j, \omega_j$ when w_{ij}'s are given that maximize the weighted likelihood
    - It is equivalent to Gaussian distribution parameter estimation when each point has a weight belonging to each distribution
      - $\mu_j = \frac{\sum_i w_{ij} x_i}{\sum_i w_{ij}}; \sigma_j^2 = \frac{\sum_i w_{ij} \left\| x_i - \mu_j \right\|^2}{\sum_i w_{ij}}; \omega_j \propto \sum_i w_{ij}$

K-Means: A Special Case of Gaussian Mixture Model

- When each Gaussian component with covariance matrix $\sigma^2 I$
  - Soft K-means
  - $p(x_i | \mu_j, \sigma^2) \propto \exp\{-\boxed{(x_i - \mu_j)^2}/\sigma^2\}$

Distance!

- When $\sigma^2 \to 0$
  - Soft assignment becomes hard assignment
  - $w_{ij} \to 1, if\ x_i$ is closest to μ_j (why?)

Case 2: Multinomial Mixture Model

- Generative model
  - For each object:
    - Pick its distribution component: $Z \sim Multi(\omega_1, \dots, \omega_k)$
    - Sample a value from the selected distribution: $X \sim Multi(\beta_{Z1}, \beta_{Z2}, \dots, \beta_{Zm})$
- Overall likelihood function
  - $L(D|\theta) = \prod_i \sum_j \omega_j p(\boldsymbol{x}_i|\boldsymbol{\beta}_j)$
    - $\sum_j \omega_j = 1; \sum_l \beta_{jl} = 1$
  - Q: What is θ here?

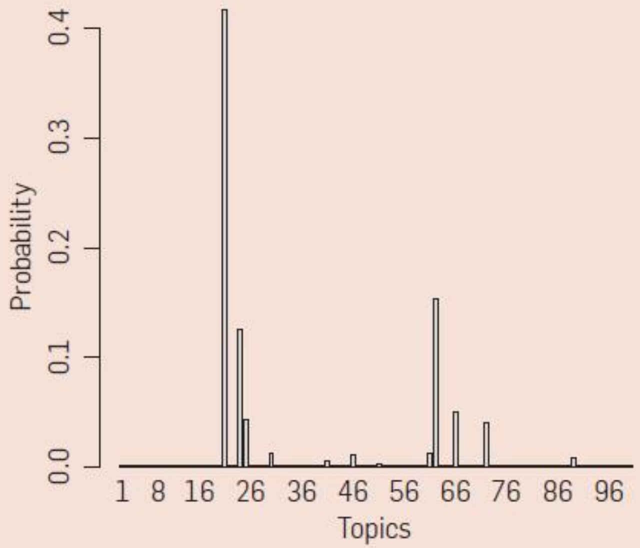
Application: Document Clustering

- A vocabulary containing m words
- Each document i:
  - A m-dimensional vector: $(c_{i1}, c_{i2}, \ldots, c_{im})$
  - c_{il} is the number of occurrence of word l appearing in document i
  - Under unigram assumption
    - $p(\boldsymbol{x}_i|\boldsymbol{\beta}_j) = \boxed{\frac{\boxed{(\sum_m c_{il})!}}{c_{i1}!\ldots c_{im}!}} \beta_{j1}^{c_{i1}} \ldots \beta_{jm}^{c_{im}}$

Length of document

Constant to all parameters

Example



"Genetics"	"Evolution"	"Disease"	"Computers"
human	evolution	disease	computer
genome	evolutionary	host	models
dna	species	bacteria	information
genetic	organisms	diseases	data
genes	life	resistance	computers
sequence	origin	bacterial	system
gene	biology	new	network
molecular	groups	strains	systems
sequencing	phylogenetic	control	model
map	living	infectious	parallel
information	diversity	malaria	methods
genetics	group	parasite	networks
mapping	new	parasites	software
project	two	united	new
sequences	common	tuberculosis	simulations

Estimating Parameters

- $l(D;\theta) = \sum_i \log \sum_j \omega_j \sum_l c_{il} log \beta_{jl}$
- Apply EM algorithm
  - E-step:
    - $\mathrm{w}_{ij} = \frac{\omega_j p(\boldsymbol{x}_i|\boldsymbol{\beta}_j)}{\sum_j \omega_j p(\boldsymbol{x}_i|\boldsymbol{\beta}_j)}$
  - M-step: maximize weighted likelihood $\sum_i w_{ij} \sum_l c_{il} log \beta_{jl}$
    - $\beta_{jl} = \boxed{\frac{\sum_i w_{ij} c_{il}}{\sum_{l'} \sum_i w_{ij} c_{il'}}}; \omega_j \propto \sum_i w_{ij}$

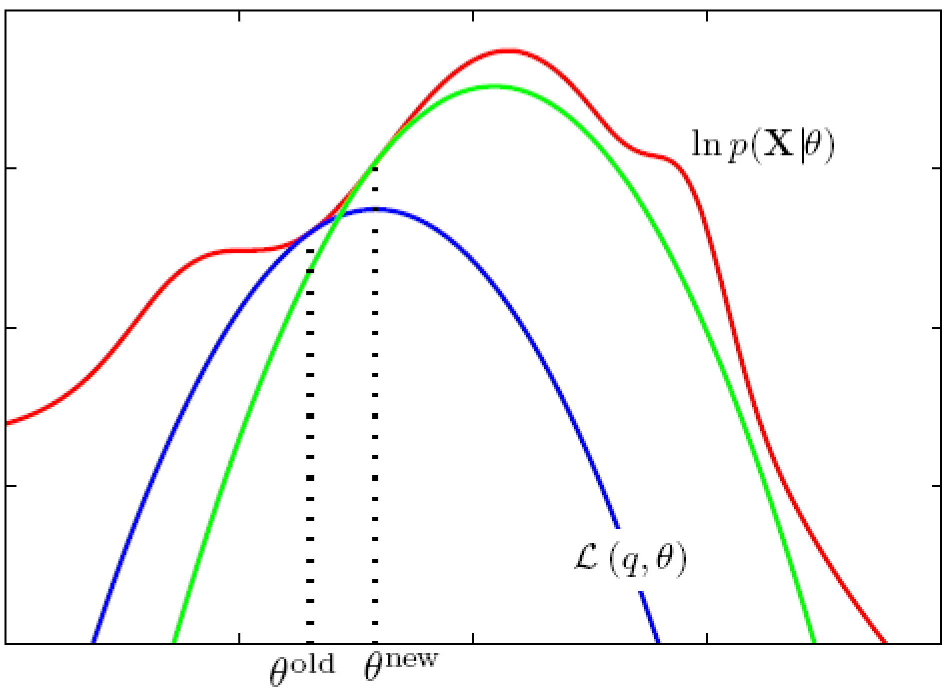
Weighted percentage of word l in cluster j

Better Way for Topic Modeling

- Topic: a word distribution
- Unigram multinomial mixture model
  - Once the topic of a document is decided, all its words are generated from that topic
- PLSA (probabilistic latent semantic analysis)
  - Every word of a document can be sampled from different topics
- LDA (Latent Dirichlet Allocation)
  - Assume priors on word distribution and/or document cluster distribution

Why EM Works?

- E-Step: computing a tight lower bound f of the original objective function at θ_{old}
- M-Step: find θ_{new} to maximize the lower bound
- $l(\theta_{new}) \geq f(\theta_{new}) \geq f(\theta_{old}) = l(\theta_{old})$



*How to Find Tight Lower Bound?

- $$\begin{aligned} \ell(\theta) &= \log \sum_h p(d,h;\theta) \\ &= \log \sum_h \frac{q(h)}{q(h)} p(d,h;\theta) \\ &= \log \sum_h q(h) \frac{p(d,h;\theta)}{q(h)} \end{aligned}$$

$q(h)$: *the tight lower bound we want to get*

- Jensen's inequality
  - $$\log \sum_h q(h) \frac{p(d,h;\theta)}{q(h)} \geq \sum_h q(h) \log \frac{p(d,h;\theta)}{q(h)}$$
- When "=" holds to get a tight lower bound?
  - $q(h) = p(h|d,\theta)$

Advantages and Disadvantages of Mixture Models

- Strength
  - Mixture models are more general than partitioning
  - Clusters can be characterized by a small number of parameters
  - The results may satisfy the statistical assumptions of the generative models
- Weakness
  - Converge to local optimal (overcome: run multi-times w. random initialization)
  - Computationally expensive if the number of distributions is large, or the data set contains very few observed data points
  - Need large data sets
  - Hard to estimate the number of clusters

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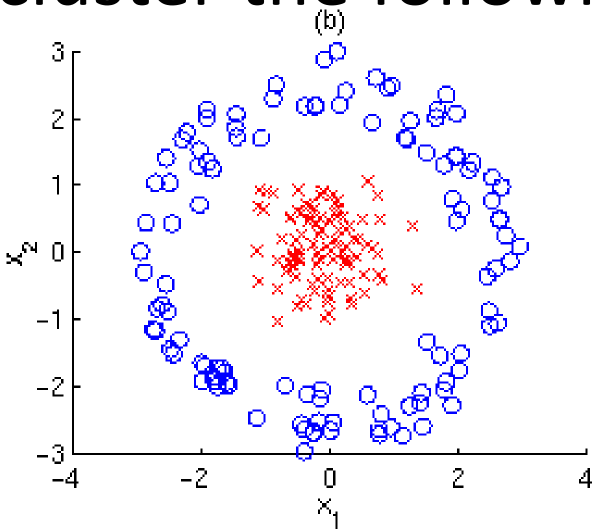
Kernel K-Means

- How to cluster the following data?



- A non-linear map: $\phi: R^n \rightarrow F$
  - Map a data point into a higher/infinite dimensional space
  - $x \rightarrow \phi(x)$
- Dot product matrix K_{ij}
  - $K_{ij} = < \phi(x_i), \phi(x_j) >$

Typical Kernel Functions

- Recall kernel SVM:

$$\textbf{Polynomial kernel of degree } h: \quad K(\boldsymbol{X_i}, \boldsymbol{X_j}) = (\boldsymbol{X_i} \cdot \boldsymbol{X_j} + 1)^h$$

$$\textbf{Gaussian radial basis function kernel}: \quad K(\boldsymbol{X_i}, \boldsymbol{X_j}) = e^{-\|\boldsymbol{X_i} - \boldsymbol{X_j}\|^2 / 2\sigma^2}$$

$$\textbf{Sigmoid kernel}: \quad K(\boldsymbol{X_i}, \boldsymbol{X_j}) = \tanh(\kappa \boldsymbol{X_i} \cdot \boldsymbol{X_j} - \delta)$$

Solution of Kernel K-Means

- Objective function under new feature space:
  - $J = \sum_{j=1}^{k} \sum_i w_{ij} ||\phi(x_i) - c_j||^2$
- Algorithm
  - By fixing assignment w_{ij}
    - $c_j = \sum_i w_{ij}\, \phi(x_i) / \sum_i w_{ij}$
  - In the assignment step, assign the data points to the closest center
    - $d(x_i, c_j) = \left|\left|\phi(x_i) - \frac{\sum_{i'} w_{i'j}\phi(x_{i'})}{\sum_{i'} w_{i'j}}\right|\right|^2 = \phi(x_i)\cdot\phi(x_i) - 2\frac{\sum_{i'} w_{i'j}\phi(x_i)\cdot\phi(x_{i'})}{\sum_{i'} w_{i'j}} + \frac{\sum_{i'}\sum_l w_{i'j}w_{lj}\phi(x_{i'})\cdot\phi(x_l)}{(\sum_{i'} w_{i'j})\text{^}2}$

$\boxed{\textbf{Do not really need to know } \phi(x), \textit{but only } K_{ij}}$

Advantages and Disadvantages of Kernel K-Means

- **Advantages**
  - Algorithm is able to identify the non-linear structures.
- **Disadvantages**
  - Number of cluster centers need to be predefined.
  - Algorithm is complex in nature and time complexity is large.
- **References**
  - Kernel k-means and Spectral Clustering by Max Welling.
  - Kernel k-means, Spectral Clustering and Normalized Cut by Inderjit S. Dhillon, Yuqiang Guan and Brian Kulis.
  - An Introduction to kernel methods by Colin Campbell.

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Summary

- Revisit k-means
  - Derivative
- Mixture models
  - Gaussian mixture model; multinomial mixture model; EM algorithm; Connection to k-means
- Kernel k-means
  - Objective function; solution; connection to k-means