Example of Modularity Maximization
\[ \mathbf{B} = \lambda \mathbf{D} \]

\[
\begin{align*}
\text{In}[2]:= & \quad \mathbf{G} = \\
\text{In}[15]:= & \quad (\mathbf{A} = \text{Normal@AdjacencyMatrix@G}) \text{ // MatrixForm} \\
\text{Out}[15]/\text{MatrixForm} = & \quad \begin{pmatrix}
0 & 2 & 1 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 \\
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 4 & 0
\end{pmatrix}
\end{align*}
\]

\[
\text{In}[5]:= \quad \mathbf{k} = \text{VertexDegree@G} \\
\text{Out}[5] = & \quad \{3, 4, 4, 5, 4\}
\]

Use \( \mathbf{K} \) instead of \( \mathbf{D} \) for that \( \mathbf{D} \) is a build-in function name

\[
\text{In}[16]:= \quad (\mathbf{K} = \text{DiagonalMatrix@k}) \text{ // MatrixForm} \\
\text{Out}[16]/\text{MatrixForm} = & \quad \begin{pmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

\[
\text{In}[7]:= \quad \mathbf{m} = \text{Total@k} / 2 \\
\text{Out}[7] = & \quad 10
\]

\[
\text{In}[17]:= \quad \mathbf{B} = \text{Table}\left[\mathbf{A[[i,j]] - \frac{k[[i]] k[[j]]}{2 m}, \{i, 5\}, \{j, 5\}\right]} \text{ // MatrixForm} \\
\text{Out}[17]/\text{MatrixForm} = & \quad \begin{pmatrix}
-9 & 7 & 2 & -3 & -3 \\
7 & -5 & -4 & 6 & -5 \\
2 & 5 & -5 & -4 & 0 \\
-3 & 4 & 0 & -5 & 5 \\
-3 & 4 & -1 & 0 & -5 \\
3 & 4 & 3 & -5 & 5
\end{pmatrix}
\]
\[ (n[24]) = N[\text{Eigensystem}[[B, K]], 5] \text{ // Grid} \]
\[
\begin{align*}
-0.93286 + & \quad 0.83355 + \quad -0.62737 + \quad -0.27332 + \quad 0 \\
0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda
\end{align*}
\]
\[ \{0.014637 + \quad -0.88933 + \quad -4.1885 + \quad 3.3965 + \quad 1.0000, \}
\]
\[ \{0.0 \times 10^{-7} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-5} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-5} \lambda, \quad 1.0000, \}
\]
\[ (-0.19606 + \quad -0.84896 + \quad 4.9576 + \quad 0.42078 + \quad 1.0000, \}
\]
\[ \{0.0 \times 10^{-6} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-5} \lambda, \quad 1.0000, \}
\]
\[ (n[25]) = \text{Timing}[N[\text{Eigensystem}[[B, K]], 5]]; \]
\[ \{10.3358, \text{Null} \} \]

\section*{As = \lambda Ds}

\[ (n[26]) = N[\text{Eigensystem}[[A, K]], 5] \text{ // Grid} \]
\[
\begin{align*}
1.0000 & \quad -0.93286 + \quad 0.83355 + \quad -0.62737 + \quad -0.27332 + \\
0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda & \quad 0 \times 10^{-6} \lambda
\end{align*}
\]
\[ \{1.0000, \quad 0.014637 + \quad -0.88933 + \quad -4.1885 + \quad 3.3965 + \}
\]
\[ \{1.0000, \quad 0 \times 10^{-7} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-5} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 1.0000, \}
\]
\[ (-0.19606 + \quad -0.84896 + \quad 4.9576 + \quad 0.42078 + \quad 1.0000, \}
\]
\[ \{1.0000, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 0 \times 10^{-5} \lambda, \quad 0 \times 10^{-6} \lambda, \quad 1.0000, \}
\]

\section*{Lu = \lambda u}

\[ (n[27]) = L = (\text{Sqrt@Inverse@K} \cdot A \cdot (\text{Sqrt@Inverse@K}) \]
\[
\begin{align*}
\{\theta, \frac{1}{\sqrt{3}}, \frac{1}{2 \sqrt{3}}, \theta, \theta\}, \{\frac{1}{\sqrt{3}}, \theta, \frac{1}{2}, \theta, \theta\}, \\
\{\frac{1}{2 \sqrt{3}}, \frac{1}{2}, \theta, \frac{1}{2 \sqrt{5}}, \theta\}, \{\theta, \theta, \frac{1}{2 \sqrt{5}}, \theta, \frac{2}{\sqrt{5}}\}, \{\theta, \theta, \theta, \frac{2}{\sqrt{5}}, \theta\}
\end{align*}
\]
In[41]= N@Eigensystem[L] // Grid

1.  -0.932863  0.83355  -0.627371  -0.273316
   {0.866025, 0.0126762, -0.770183, -3.62731, 2.94144, 1.11803, 1.}
Out[41]= 1., 1., -0.196064, -0.848963, 4.95759, 0.420775,
       1.11803, 1.}  0.351165, -0.525976, -2.03203, -3.62649,
       -1.04297, 1.}  0.931937, 1.}  -0.701422, 1.}  -0.305577, 1.}
Result

```
In[42]:= CommunityGraphPlot[G, {{1, 2, 3}, {4, 5}}]
```

Out[42]=

![Graph Plot](image)