# CS249: SPECIAL TOPICS MINING INFORMATION/SOCIAL NETWORKS 

## Overview of Networks

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## Overview of Information Network

 Analysis- Network Representation
- Network Properties
- Network Generative Models
- Random Walk and Its Applications


## Networks Are Everywhere



Aspirin


from H. Jeong et al Nature 411, 41 (2001)
Yeast protein interaction network


## Representation of a Network: Graph

- $G=<V, E>$
- $V=\left\{u_{1}, \ldots, u_{n}\right\}$ : node set
- $E \subseteq V \times V$ : edge set
- Adjacency matrix
- $A=\left\{a_{i j}\right\}, i, j=1, \ldots, N$
- $a_{i j}=1$, if $<u_{i}, u_{j}>\in E$
- $a_{i j}=0$, if $<u_{i}, u_{j}>\notin E$
- Network types
- Undirected graph vs. Directed graph
- $A=A^{\mathrm{T}}$ vs. $A \neq A^{\mathrm{T}}$
- Binary graph Vs. Weighted graph
- Use $W$ instead of $A$, where $w_{i j}$ represents the weight of edge $<u_{i}, u_{j}>$


## Example




Adjacency matrix A

## Degree of Nodes

- Let a network G = (V, E)
- Undirected Network

- Degree (or degree centrality) of a vertex: $d\left(v_{i}\right)$
- \# of edges connected to it, e.g., $d(A)=4, d(H)=2$
- Directed network
- In-degree of a vertex $\mathrm{d}_{\mathrm{in}}\left(\mathrm{v}_{\mathrm{i}}\right)$ :
- \# of edges pointing to $v_{i}$
- E.g., $d_{\text {in }}(A)=3, d_{i n}(B)=2$
- Out-degree of a vertex $d_{\text {out }}\left(v_{i}\right)$ :
- \# of edges from $v_{i}$
- E.g., $d_{\text {out }}(A)=1, d_{\text {out }}(B)=2$



## Degree Distribution



Graph $\mathrm{G}_{1}$

- Degree sequence of a graph: The list of degrees of the nodes sorted in non-increasing order
- E.g., in $\mathrm{G}_{1}$, degree sequence: (4, 3, 2, 2, 1)
- Degree frequency distribution of a graph: Let $N_{k}$ denote the \# of vertices with degree $k$
- $\left(\mathrm{N}_{0}, \mathrm{~N}_{1}, \ldots, \mathrm{~N}_{\mathrm{t}}\right)$, t is max degree for a node in G
- E.g., in $\mathrm{G}_{1}$, degree frequency distribution: ( $0,1,2,1,1$ )
- Degree distribution of a graph:

Probability mass function $f$ for random variable $X$

- $\left(f(0), f(1), \ldots, f(t)\right.$, where $f(k)=P(X=k)=N_{k} / n$
- E.g., in $\mathrm{G}_{1}$, degree distrib.: ( $0,0.2,0.4,0.2,0.2$ )


## Path

- Path: A sequence of vertices that every consecutive pair of vertices in the sequence is connected by an edge in the network
- Length of a path: \# of edges traversed along the path
- Total \# of path of length 2 from $j$ to $i$, via any vertex in $N_{i j}{ }^{(2)}$ is

$$
N_{i j}^{(2)}=\sum_{k=1}^{n} A_{i k} A_{k j}=\left[A^{2}\right]_{i j}
$$

- Generalizing to path of arbitrary length, we have:

$$
N_{i j}{ }^{(r)}=\left[A^{r}\right]_{i j}
$$

## Radius and Diameter



- Eccentricity: The eccentricity of a node $v_{i}$ is the maximum distance from $v_{i}$ to any other nodes in the graph
- $\mathrm{e}\left(\mathrm{v}_{\mathrm{i}}\right)=\max _{\mathrm{j}}\left\{\mathrm{d}\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{j}}\right)\right\}$
- E.g., $e(A)=1, e(F)=e(B)=e(D)=e(H)=2$
- Radius of a connected graph G: the min eccentricity of any node in G
- $\mathrm{r}(\mathrm{G})=\min _{\mathrm{i}}\left\{\mathrm{e}\left(\mathrm{v}_{\mathrm{i}}\right)\right\}=\min _{\mathrm{i}}\left\{\max _{\mathrm{j}}\left\{\mathrm{d}\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{j}}\right)\right\}\right\}$
- E.g., $r\left(G_{1}\right)=1$
- Diameter of a connected graph G: the max eccentricity of any node in G
- $\mathrm{d}(\mathrm{G})=\max _{\mathrm{i}}\left\{\mathrm{e}\left(\mathrm{v}_{\mathrm{i}}\right)\right\}=\max _{\mathrm{i}, \mathrm{j}}\left\{\mathrm{d}\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{j}}\right)\right\}$
- E.g., $d\left(G_{1}\right)=2$
- Diameter is sensitive to outliers. Effective diameter: min \# of hops for which a large fraction, typically $90 \%$, of all connected pairs of nodes can reach each other


## Clustering Coefficient

- Real networks are sparse: Corresponding to a complete graph
- Clustering coefficient of a node $v_{i}$ : A measure of the density of edges in the neighborhood of $v_{i}$
- Let $G_{i}=\left(V_{i}, E_{i}\right)$ be the subgraph induced by the neighbors of vertex $v_{i},\left|V_{i}\right|$ $=n_{i}\left(\#\right.$ of neighbors of $\left.v_{i}\right)$, and $\left|E_{i}\right|=m_{i}(\#$ of edges among the neighbors of $v_{i}$ )
- Clustering coefficient of $v_{i}$ for undirected network is

$$
C\left(v_{i}\right)=\frac{\# \text { edges in } G_{i}}{\max \# \text { edges in } G_{i}}=\frac{m_{i}}{\binom{n_{i}}{2}}=\frac{2 \times m_{i}}{n_{i}\left(n_{i}-1\right)}
$$

- For directed network, $C\left(v_{i}\right)=\frac{\# \text { edges in } G_{i}}{\text { max \# edges in } G_{i}}=\frac{m_{i}}{n_{i}\left(n_{i}-1\right)}$
- Clustering coefficient of a graph G: $\quad C(G)=\frac{1}{n} \sum C\left(v_{i}\right)$
- Averaging the local clustering coefficient of all the vertices (Watts \& Strogatz)


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## More Than a Graph

## - A typical network has the following common

 properties:- Few connected components:
- often only 1 or a small number, independent of network size
- Small diameter:
- often a constant independent of network size (like 6)
- growing only logarithmically with network size or even shrink?
- A high degree of clustering:
- considerably more so than for a random network
- A heavy-tailed degree distribution:
- a small but reliable number of high-degree vertices
- often of power law form


## Sparse

- For complete Graph
- Average degree: N
- For real-world network
- Average degree: $\langle k\rangle=2 E / N \ll N$


## Small World Property

- Small world phenomenon (Six degrees of separation)
- Stanley Milgram's experiments (1960s)
- Microsoft Instant Messaging (IM) experiment: J. Leskovec \& E. Horvitz (WWW’08)
- 240 M active user accounts: Est. avg. distance 6.6 \& est. mean median 7
- Why small world?
$\cdot N(d) \approx 1+\langle k\rangle+\langle k\rangle^{2}+\cdots+\langle k\rangle^{d}=\frac{\langle k\rangle^{d+1}-1}{\langle k\rangle-1} \approx\langle k\rangle^{d}$
- E.g., $d \approx \frac{\ln N}{\ln \langle k\rangle} \approx \frac{\ln \left(7 \times 10^{9}\right)}{\ln \left(10^{3}\right)} \approx 3.28$


## Degree Distribution: Power Law

a.


From Barabasi 2016

$$
f(k) \propto k^{-\gamma}
$$

Typically $0<\gamma<2$; smaller $\gamma$ gives heavier tail



The degree distribution of the (a) Internet, (b) science collaboration network, and (c) protein interaction network

## High Clustering Coefficient

- Clustering effect: a high clustering coefficient for graph G
- Friends' friends are likely friends.
- A lot of triangles
- C(k): avg clustering coefficient for nodes with degree k


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## Network Generative Models

- All of the network generation models we will study are probabilistic or statistical in nature
- They can generate networks of any size
- They often have various parameters that can be set:
- size of network generated
- average degree of a vertex
- fraction of long-distance connections
- The models generate a distribution over networks
- Statements are always statistical in nature:
- with high probability, diameter is small
- on average, degree distribution has heavy tail


## Examples

## - Erdös-Rényi Random graph model:

- Gives few components and small diameter
- does not give high clustering and heavy-tailed degree distributions
- is the mathematically most well-studied and understood model
- Watts-Strogatz small world graph model:
- gives few components, small diameter and high clustering
- does not give heavy-tailed degree distributions
- Barabási-Albert Scale-free model:
- gives few components, small diameter and heavy-tailed distribution
- does not give high clustering
- Stochastic Block Model


## Erdös-Rényi (ER) Random Graph Model

## - Every possible edge occurs independently with probability $p$

- $G(N, p)$ : a network of $\mathbf{N}$ nodes, each node pair is connected with probability of $p$
- Paul Erdős and Alfréd Rényi: "On Random Graphs" (1959)
-E. N. Gilbert: "Random Graphs" (1959) (proposed independently)
- Usually, $N$ is large and $p \sim 1 / N$
- Choices: $p=1 / 2 N, p=1 / N, p=2 / N, p=10 / N, p=\log (N) / N$, etc.


## Degree Distribution

- The degree distribution of a random (small) network follows binomial distribution

$$
p_{k}=\binom{N-1}{k} p^{k}(1-p)^{N-1-k}
$$

- When N is large and Np is fixed, approximated by Poisson distributi

$$
p_{k}=e^{-\langle k\rangle} \frac{\langle k\rangle^{k}}{k!}
$$



From Barabasi 2016

## Watts-Strogatz small world model

- Interpolates between regular lattice and a random network to generate graphs with
- Small-world: short average path lengths
- High clustering coefficient:


$p:$ the prob. each link is rewired
to a randomly chosen node
$C(p)$ : clustering coeff.
$L(p)$ : average path length


## Barabási-Albert Model: Preferential Attachment

- Major limitation of the Watts-Strogatz model
- It produces graphs that are homogeneous in degree
- Real networks are often inhomogeneous in degree, having hubs and a scale-free degree distribution (scale-free networks)
- Scale-free networks are better described by the preferential attachment family of models, e.g., the Barabási-Albert (BA) model
- "rich-get-richer": New edges are more likely to link to nodes with higher degrees
- Preferential attachment: The probability of connecting to a node is proportional to the current degree of that node
- This leads to the proposal of a new model: scale-free network, a network whose degree distribution follows a power law, at least asymptotically


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## The History of PageRank

- PageRank was developed by Larry Page (hence the name Page-Rank) and Sergey Brin.
- It is first as part of a research project about a new kind of search engine. That project started in 1995 and led to a functional prototype in 1998.
- Shortly after, Page and Brin founded Google.


## Ranking web pages

- Web pages are not equally "important"
- www.cnn.com vs. a personal webpage
- Inlinks as votes
- The more inlinks, the more important
- Are all inlinks equal?
- Higher ranked inlink should play a more important role
- Recursive question!


## Simple recursive formulation

- Each link's vote is proportional to the importance of its source page
- If page $P$ with importance $x$ has $n$ outlinks, each link gets $x / n$ votes
- Page P's own importance is the sum of the votes on its inlinks



## Matrix formulation

- Matrix $\mathbf{M}$ has one row and one column for each web page

| - Suppose page j has n outlinks | y | 1 | 1 | 0 |
| :--- | :--- | :--- | :--- | :--- |
| - If $\mathrm{j}->\mathrm{i}$, then $\mathrm{M}_{\mathrm{ij}}=1 / \mathrm{n}$ | a | 1 | 0 | 1 |
| - Else $\mathrm{M}_{\mathrm{ij}}=0$ | m | 0 | 1 | 0 |

- $\mathbf{M}$ is a column stochastic matrix
- Columns sum to 1
- Suppose $r$ is a vector with one entry per web page
- $r_{i}$ is the importance score of page $i$
- Call it the rank vector
- $|\mathbf{r}|=1$ (i.e., $r_{1}+r_{2}+\cdots+r_{N}=1$ )


## Eigenvector formulation

- The flow equations can be written

$$
r=M r
$$

- So the rank vector is an eigenvector of the stochastic web matrix
- In fact, its first or principal eigenvector, with corresponding eigenvalue 1


## Example



$$
\begin{aligned}
& y=y / 2+a / 2 \\
& a=y / 2+m
\end{aligned}
$$

$$
m=a / 2
$$




| y |
| :--- |
| a |
| m |$=$| $1 / 2$ | $1 / 2$ | 0 |  |
| :---: | :---: | :---: | :---: |
| $1 / 2$ | 0 | 1 |  |
| 0 | $1 / 2$ | 0 | y |
| a |  |  |  |
| m |  |  |  |

## Power Iteration method

- Simple iterative scheme
- Suppose there are N web pages
- Initialize: $\mathbf{r}^{0}=[1 / \mathrm{N}, \ldots, 1 / \mathrm{N}]^{\top}$
- Iterate: $\mathbf{r}^{\mathbf{k}+1}=\mathbf{M r} \mathbf{N}^{\mathbf{k}}$
- Stop when $\left|\mathbf{r}^{k+1}-\mathbf{r}^{\mathrm{k}}\right|_{1}<\varepsilon$
- $|\mathbf{x}|_{1}=\sum_{1 \leq i \leq N}\left|x_{i}\right|$ is the $L_{1}$ norm
- Can use any other vector norm e.g., Euclidean


## Power Iteration Example



| y |  |  |  |  |  |  |
| :--- | ---: | ---: | :---: | :--- | :--- | ---: |
| a |  |  |  |  |  |  |
| m | $=\quad 1 / 3$ | $1 / 3$ | $5 / 12$ | $3 / 8$ |  | $2 / 5$ |
| $1 / 3$ | $1 / 2$ | $1 / 3$ | $11 / 24$ | $\ldots$ | $2 / 5$ |  |
| $1 / 3$ | $1 / 6$ | $1 / 4$ | $1 / 6$ |  | $1 / 5$ |  |
|  | $r_{0}$ | $r_{1}$ | $r_{2}$ | $r_{3}$ | $\ldots$ | $r^{*}$ |

## Random Walk Interpretation

- Imagine a random web surfer
- At any time $t$, surfer is on some page $P$
- At time $t+1$, the surfer follows an outlink from $P$ uniformly at random
- Ends up on some page Q linked from P
- Process repeats indefinitely
- Let $\mathbf{p}(\mathrm{t})$ be a vector whose $\mathrm{i}^{\text {th }}$ component is the probability that the surfer is at page $i$ at time $t$
- $\mathbf{p}(\mathrm{t})$ is a probability distribution on pages


## The stationary distribution

- Where is the surfer at time $t+1$ ?
- Follows a link uniformly at random
- $\mathbf{p}(\mathrm{t}+1)=\mathbf{M p}(\mathrm{t})$
- Suppose the random walk reaches a state such that $\mathbf{p}(\mathrm{t}+1)=\mathbf{M p}(\mathrm{t})=\mathbf{p}(\mathrm{t})$
- Then $\mathbf{p}(\mathrm{t})$ is called a stationary distribution for the random walk
- Our rank vector $\mathbf{r}$ satisfies $\mathbf{r}=\mathbf{M r}$
- So it is a stationary distribution for the random surfer


## Existence and Uniqueness

A central result from the theory of random walks (aka Markov processes):

For graphs that satisfy certain conditions, the stationary distribution is unique and eventually will be reached no matter what the initial probability distribution at time $t=0$.

## Spider traps

- A group of pages is a spider trap if there are no links from within the group to outside the group
- Random surfer gets trapped
- Spider traps violate the conditions needed for the random walk theorem


## Microsoft becomes a spider trap



## Random teleports

- The Google solution for spider traps
- At each time step, the random surfer has two options:
- With probability $\beta$, follow a link at random
- With probability $1-\beta$, jump to some page uniformly at random
- Common values for $\beta$ are in the range 0.8 to 0.9
- Surfer will teleport out of spider trap within a few time steps


## Random teleports $(\beta=0.8)$


$\rightarrow-->$ : teleport links from "Yahoo"

|  | $7 / 15$ | $7 / 15$ | $1 / 15$ |
| :--- | :--- | :--- | :--- |
| a | $7 / 15$ | $1 / 15$ | $1 / 15$ |
| m | $1 / 15$ | $7 / 15$ | $13 / 15$ |
|  |  |  |  |

## Random teleports ( $\beta=0.8$ )



## Matrix formulation

- Suppose there are N pages
- Consider a page j, with set of outlinks $\mathrm{O}(\mathrm{j})$
- We have $M_{i j}=1 /|O(j)|$ when $j->i$ and $M_{i j}=0$ otherwise
- The random teleport is equivalent to
- adding a teleport link from $j$ to every other page with probability (1- $\beta$ )/N
- reducing the probability of following each outlink from $1 /|O(j)|$ to $\beta /|O(j)|$
- Equivalent: tax each page a fraction (1- $\beta$ ) of its score and redistribute evenly


## PageRank

- Construct the $\mathrm{N}-\mathrm{by}-\mathrm{N}$ matrix A as follows
- $A_{i j}=\beta M_{i j}+(1-\beta) / N$
- Verify that $\mathbf{A}$ is a stochastic matrix
- The page rank vector $\mathbf{r}$ is the principal eigenvector of this matrix
- satisfying r = Ar
- Equivalently, $r$ is the stationary distribution of the random walk with teleports


## Dead ends

- Pages with no outlinks are "dead ends" for the random surfer
- Nowhere to go on next step


## Microsoft becomes a dead end



$$
0.8 \begin{array}{|ccc|}
\hline 1 / 2 & 1 / 2 & 0 \\
1 / 2 & 0 & 0 \\
0 & 1 / 2 & 0
\end{array} \quad+0.2 \begin{array}{ccc}
1 / 3 & 1 / 3 & 1 / 3 \\
1 / 3 & 1 / 3 & 1 / 3 \\
1 / 3 & 1 / 3 & 1 / 3
\end{array}
$$



## Dealing with dead-ends

- Teleport
- Follow random teleport links with probability 1.0 from dead-ends
- Adjust matrix accordingly
- Prune and propagate
- Preprocess the graph to eliminate dead-ends
- Might require multiple passes
- Compute page rank on reduced graph
- Approximate values for deadends by propagating values from reduced graph


## Dealing dead end: teleport



## Dealing dead end: reduce graph



## Computing PageRank

- Key step is matrix-vector multiplication
- $\mathbf{r}^{\text {new }}=\mathbf{A r}{ }^{\text {old }}$
- Easy if we have enough main memory to hold A, $\mathbf{r}^{\text {old }}, \mathbf{r}^{\text {new }}$
- Say N = 1 billion pages
- We need 4 bytes for each entry (say)
- 2 billion entries for vectors, approx 8GB
- Matrix A has $\mathrm{N}^{2}$ entries
- $10^{18}$ is a large number!


## Rearranging the equation

## $r=A r$, where

$$
\begin{aligned}
A_{i j} & =\beta M_{i j}+(1-\beta) / N \\
r_{i} & =\sum_{1 \leq j \leq N} A_{i j} r_{j} \\
r_{i} & =\sum_{1 \leq j \leq N}\left[\beta M_{i j}+(1-\beta) / N\right] r_{j} \\
& =\beta \sum_{1 \leq j \leq N} M_{i j} r_{j}+(1-\beta) / N \sum_{1 \leq j \leq N} r_{j} \\
& =\beta \sum_{1 \leq j \leq N} M_{i j} r_{j}+(1-\beta) / N, \text { since }|r|=1 \\
r & =\beta M r+[(1-\beta) / N]_{N}
\end{aligned}
$$

where $[x]_{N}$ is an N -vector with all entries x

## Sparse matrix formulation

- We can rearrange the page rank equation:
- $\mathbf{r}=\beta \mathbf{M r}+[(1-\beta) / \mathrm{N}]_{N}$
- $[(1-\beta) / \mathrm{N}]_{N}$ is an N -vector with all entries $(1-\beta) / \mathrm{N}$
- $\mathbf{M}$ is a sparse matrix!
- 10 links per node, approx 10 N entries
- So in each iteration, we need to:
- Compute $\mathbf{r}^{\text {new }}=\beta$ Mrold
- Add a constant value $(1-\beta) / \mathrm{N}$ to each entry in $\mathbf{r}^{\text {new }}$


## Sparse matrix encoding

- Encode sparse matrix using only nonzero entries
- Space proportional roughly to number of links
- say 10 N , or $4^{*} 10^{*} 1$ billion $=40 \mathrm{~GB}$
- still won't fit in memory, but will fit on disk

| source |
| :--- |
| node |


| 0 | 3 | $1,5,7$ |
| :--- | :--- | :--- |
| 1 | 5 | $17,64,113,117,245$ |
| 2 | 2 | 13,23 |

## Basic Algorithm

- Assume we have enough RAM to fit $\mathbf{r}^{\text {new }}$, plus some working memory
- Store rold and matrix M on disk


## Basic Algorithm:

- $\quad$ Initialize: $r^{\text {old }}=[1 / \mathrm{N}]_{N}$
- Iterate:
- Update: Perform a sequential scan of $\mathbf{M}$ and $\mathbf{r}^{\text {rod }}$ to update $\mathbf{r}^{\text {new }}$
- Write out $\mathbf{r}^{\text {new }}$ to disk as rold for next iteration
- Every few iterations, compute $\mid \mathbf{r}^{\text {new }} \mathbf{- r} \mathbf{r o l d}^{\text {old }}$ and stop if it is below threshold
- Need to read in both vectors into memory


## Summary

- Network Representation
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## Paper Sign-Up

-https://docs.google.com/spreadsheets/d/1Sao PGP2SsYyaycX82T7mF efbiueOl53bnZtZSO4Bt Q/edit?usp=sharing

- If you are still on waiting list
- Sign-up for Presenter 4 only


## Credits

- This is 4-credit course, please change it if you are current enrolled with 2-credit


## Course Project Examples

- Citation graph summary
- Find k papers that can tell the main structure evolution of a certain field
- Name disambiguation problem in DBLP
- Different people may share the same name, e.g., distinguish "Wei Wang"'s;
- Same person may have different forms of names, e.g., initials, middle names, typos
- User profile prediction in heterogeneous information networks
- Suppose we only know small number of labels for people's ideology, profession, education, can we predict the remaining?
- Sentence embedding
- Can we find the most similar sentences or S-VO (subject-verb-object) triplets to the given one, by converting the text into a network?

