

CS247: ADVANCED DATA MINING

02: Probabilistic Classifiers

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Methods to Learn

	Vector Data	Text Data	Graph & Network	Recommender Systems
Classification	Naïve Bayes; Logistic Regression; NN		Label Propagation	
Clustering	K-means; kernel k-means; Mixture Models	PLSA; LDA	Spectral Clustering	Matrix Factorization
Prediction	NN			Collaborative Filtering; Factorization machine; Hybrid CF; Recommendation with graph regularization
Ranking			PageRank	
Similarity Search			P-PageRank	
Representation Learning		Word embedding	Network embedding	Deep collaborative learning


Content

- Probabilistic Models 
- Naïve Bayes
- Logistic Regression
- Generative Models and Discriminative Models
- Summary


Probabilistic Models for Classification

- Data: $D = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$
 - A data point (\mathbf{x}_i, y_i) contains a feature vector and a label
 - n : number of data points
- Model: $p(D|\theta)$
 - E.g., under I.I.D. assumption
 - $p(D|\theta) = \prod_i p(\mathbf{x}_i, y_i|\theta)$ (if modeling joint distribution)
 - $p(D|\theta) = \prod_i p(y_i|\mathbf{x}_i, \theta)$ (if modeling conditional distribution, conditional i.i.d.)
- Inference: query the model
 - *e.g., under I.I.D assumption, $p(y_i|\mathbf{x}_i, \theta) = ?$*
- Learning: $\hat{\theta} = ?$

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Naïve Bayes for Text

- Text Data 
- Revisit of Multinomial Distribution
- Multinomial Naïve Bayes

Text Data

- Word/term
- Document
 - A sequence of words
- Corpus
 - A collection of documents



Text Classification Applications

- Spam detection

From: airak@medicana.com.tr

Subject: Loan Offer

Do you need a personal or business loan urgent that can be process within 2 to 3 working days? Have you been frustrated so many times by your banks and other loan firm and you don't know what to do? Here comes the Good news Deutsche Bank Financial Business and Home Loan is here to offer you any kind of loan you need at an affordable interest rate of 3% If you are interested let us know.

- Sentiment analysis



The Lion King, complete with jaunty songs by Elton John and Tim Rice, is undeniably and fully worthy of its glorious Disney heritage. It is a gorgeous triumph -- one lion in which the studio can take justified pride.



Between traumas, the movie serves up soothingly banal musical numbers (composed by Elton John and Tim Rice) and silly, rambunctious comedy.

July 31, 2013 | [Full Review...](#)


Represent A Document

Ex: “Do you need a personal or business loan urgent?”

- A document d is represented by a sequence of words selected from a **vocabulary**
 - $\mathbf{w}_d = (w_{d1}, w_{d2}, \dots, w_{dN_d})$, where w_{di} is the id of i -th word in document d and N_d is the length of document d
- A bag-of-words representation
 - $\mathbf{x}_d = (x_{d1}, x_{d2}, \dots, x_{dN})$, where x_{dn} is the number of words for n th word in the vocabulary
 - $x_{dn} = \sum_i 1(w_{di} == n)$

Example


- c1: *Human machine interface* for Lab ABC computer applications
 c2: A survey of user opinion of computer system response time
 c3: The *EPS user interface* management system
 c4: System and human system engineering testing of *EPS*
 c5: Relation of user-perceived response time to error measurement
- m1: The generation of random, binary, unordered trees
 m2: The intersection graph of paths in trees
 m3: *Graph minors IV*: Widths of trees and well-quasi-ordering
 m4: *Graph minors*: A survey



	c1	c2	c3	c4	c5	m1	m2	m3	m4
<i>human</i>	1	0	0	1	0	0	0	0	0
<i>interface</i>	1	0	1	0	0	0	0	0	0
<i>computer</i>	1	1	0	0	0	0	0	0	0
<i>user</i>	0	1	1	0	1	0	0	0	0
<i>system</i>	0	1	1	2	0	0	0	0	0
<i>response</i>	0	1	0	0	1	0	0	0	0
<i>time</i>	0	1	0	0	1	0	0	0	0
<i>EPS</i>	0	0	1	1	0	0	0	0	0
<i>survey</i>	0	1	0	0	0	0	0	0	1
<i>trees</i>	0	0	0	0	0	1	1	1	0
<i>graph</i>	0	0	0	0	0	0	1	1	1
<i>minors</i>	0	0	0	0	0	0	0	1	1

x_d

Naïve Bayes for Text

- Text Data
- Revisit of Multinomial Distribution 
- Multinomial Naïve Bayes

Bernoulli and Categorical Distribution

- Bernoulli distribution
 - Discrete distribution that takes two values $\{0,1\}$
 - $P(X = 1) = p$ and $P(X = 0) = 1 - p$
 - E.g., toss a coin with head and tail
- Categorical distribution
 - Discrete distribution that takes more than two values, i.e., $x \in \{1, \dots, K\}$
 - Also called generalized Bernoulli distribution, multinoulli distribution
 - $P(X = k) = p_k$ and $\sum_k p_k = 1$
 - E.g., get 1-6 from a dice with $1/6$



Binomial and Multinomial Distribution

- Binomial distribution

- Number of successes (i.e., total number of 1's) by repeating n trials of independent Bernoulli distribution with probability p

- x : number of successes

- $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$


- Multinomial distribution (multivariate random variable)

- Repeat n trials of independent categorical distribution

- Let x_k be the number of times value k has been observed, note $\sum_k x_k = n$

- $P(X_1 = x_1, X_2 = x_2, \dots, X_K = x_K) = \frac{n!}{x_1! x_2! \dots x_K!} \prod_k p_k^{x_k}$

Naïve Bayes for Text

- Text Data
- Revisit of Multinomial Distribution
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Now Come to Text Setting: Modeling

- Model $p(\mathbf{x}_d, y_d) = p(x_d | y_d) p(y_d)$
 - Model $p(\mathbf{w}_d | y_d = j)$ or $p(\mathbf{x}_d | y_d = j)$ for class j
 - Each word in the sequence w_{di} is sampled from categorical distribution with parameter vector $\boldsymbol{\beta}_j = (\beta_{j1}, \beta_{j2}, \dots, \beta_{jN})$ independently
 - $p(w_{di} | y_d = j) = \beta_{jw_{di}}$ and $p(\mathbf{w}_d | y_d = j) = \prod_i \beta_{jw_{di}} = \prod_n \beta_{jn}^{x_{dn}}$
 - Where x_{dn} is the number of words for n th word in the vocabulary
 - Model $p(y_d = j)$
 - Follow categorical distribution with parameter vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_m)$, i.e.,
 - $p(y_d = j) = \pi_j$

Classification Process Assuming Parameters are Given: Inference

- Find y that maximizes $p(y_d | \mathbf{x}_d)$, which is equivalently to maximize

$$\begin{aligned} y_d^* &= \operatorname{argmax}_j p(\mathbf{x}_d, y_d = j) \\ &= \operatorname{argmax}_j p(\mathbf{x}_d | y_d = j) p(y_d = j) \\ &= \operatorname{argmax}_j \prod_n \beta_{jn}^{x_{dn}} \times \pi_j \\ &= \operatorname{argmax}_j \sum_n x_{dn} \log \beta_{jn} + \log \pi_j \end{aligned}$$

Parameter Estimation via MLE: Learning

- Given a corpus and labels for each document

- $D = \{(\mathbf{x}_d, y_d)\}$

- Find the MLE estimators for $\Theta = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_m, \boldsymbol{\pi})$

- The log likelihood function for the training dataset

$$\begin{aligned} \log L(\Theta) &= \log \prod_d p(\mathbf{x}_d, y_d | \Theta) = \sum_d \log p(\mathbf{x}_d, y_d | \Theta) \\ &= \sum_d \log [p(\mathbf{x}_d | y_d) p(y_d)] = \sum_d \left(\sum_n x_{dn} \log \beta_{y_d n} + \log \pi_{y_d} \right) \end{aligned}$$

- The optimization problem

$$\max_{\Theta} \log L(\Theta)$$

s. t.

$$\pi_j \geq 0 \text{ and } \sum_j \pi_j = 1$$

$$\beta_{jn} \geq 0 \text{ and } \sum_n \beta_{jn} = 1 \text{ for all } j$$

Solve the Optimization Problem

- Use the Lagrange multiplier method
- Solution

- $$\hat{\beta}_{jn} = \frac{\sum_{d:y_d=j} x_{dn}}{\sum_{d:y_d=j} \sum_{n'} x_{dn'}}$$

- $\sum_{d:y_d=j} x_{dn}$: total count of word n in class j

- $\sum_{d:y_d=j} \sum_{n'} x_{dn'}$: total count of words in class j

- $$\hat{\pi}_j = \frac{\sum_d 1(y_d=j)}{|D|}$$

- $1(y_d = j)$ is the indicator function, which equals to 1 if $y_d = j$ holds

- $|D|$: total number of documents

Smoothing

- What if some word n does not appear in some class j in training dataset?
 - $\hat{\beta}_{jn} = \frac{\sum_{d:y_d=j} x_{dn}}{\sum_{d:y_d=j} \sum_{n'} x_{dn'}} = 0$
 - $\Rightarrow p(\mathbf{x}_d | y_d = j) = \prod_n \beta_{jn}^{x_{dn}} = 0$
 - But other words may have a strong indication the document belongs to class j
- Solution: add-1 smoothing or Laplace smoothing
 - $\hat{\beta}_{jn} = \frac{\sum_{d:y_d=j} x_{dn} + 1}{\sum_{d:y_d=j} \sum_{n'} x_{dn'} + N}$
 - N : total number of words in the vocabulary
 - Check: $\sum_n \hat{\beta}_{jn} = 1$?

Example

- Data:

	Doc	Words	Class
Training	1	Chinese Beijing Chinese	c
	2	Chinese Chinese Shanghai	c
	3	Chinese Macao	c
	4	Tokyo Japan Chinese	j
Test	5	Chinese Chinese Chinese Tokyo Japan	?

- Vocabulary:

Index	1	2	3	4	5	6
Word	Chinese	Beijing	Shanghai	Macao	Tokyo	Japan

- Learned parameters (with smoothing):

$$\hat{\beta}_{c1} = \frac{5+1}{8+6} = \frac{3}{7}$$

$$\hat{\beta}_{c2} = \frac{1+1}{8+6} = \frac{1}{7}$$

$$\hat{\beta}_{c3} = \frac{1+1}{8+6} = \frac{1}{7}$$

$$\hat{\beta}_{c4} = \frac{1+1}{8+6} = \frac{1}{7}$$

$$\hat{\beta}_{c5} = \frac{0+1}{8+6} = \frac{1}{14}$$

$$\hat{\beta}_{c6} = \frac{0+1}{8+6} = \frac{1}{14}$$

$$\hat{\beta}_{j1} = \frac{1+1}{3+6} = \frac{2}{9}$$

$$\hat{\beta}_{j2} = \frac{0+1}{3+6} = \frac{1}{9}$$

$$\hat{\beta}_{j3} = \frac{0+1}{3+6} = \frac{1}{9}$$

$$\hat{\beta}_{j4} = \frac{0+1}{3+6} = \frac{1}{9}$$

$$\hat{\beta}_{j5} = \frac{1+1}{3+6} = \frac{2}{9}$$

$$\hat{\beta}_{j6} = \frac{1+1}{3+6} = \frac{2}{9}$$

$$\hat{\pi}_c = \frac{3}{4}$$

$$\hat{\pi}_j = \frac{1}{4}$$

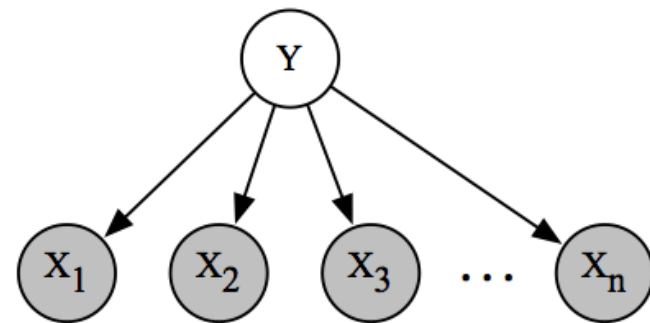
Example (Continued)

- Classification stage
 - For the test document $d=5$, compute
 - $p(y_5 = c | \mathbf{x}_5) \propto p(y_5 = c) \times \prod_n \beta_{cn}^{x_{5n}} = \frac{3}{4} \times \left(\frac{3}{7}\right)^3 \times \left(\frac{1}{14}\right) \times \left(\frac{1}{14}\right) \approx 0.0003$
 - $p(y_5 = j | \mathbf{x}_5) \propto p(y_5 = j) \times \prod_n \beta_{jn}^{x_{5n}} = \frac{1}{4} \times \left(\frac{2}{9}\right)^3 \times \left(\frac{2}{9}\right) \times \left(\frac{2}{9}\right) \approx 0.0001$
 - Conclusion: \mathbf{x}_5 should be classified into c class


A More General Naïve Bayes Framework

- Let D be a training set of tuples and their class labels, and each tuple is represented by an p -D attribute vector $\mathbf{x} = (x_1, x_2, \dots, x_p)$
- Suppose there are m classes $y \in \{1, 2, \dots, m\}$
- Goal: Find $y = \arg \max_y p(y|\mathbf{x}) = p(y, \mathbf{x})/p(\mathbf{x}) \propto p(\mathbf{x}|y)p(y)$
- A simplified assumption: attributes are **conditionally independent given the class** (class conditional independency):

- $p(\mathbf{x}|y) = \prod_k p(x_k|y)$
- $p(x_k|y)$ can follow any distribution,
 - e.g., Gaussian, Bernoulli, categorical, ...



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Data

- Data: $D = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$
 - A data point (\mathbf{x}_i, y_i) contains a feature vector and a label
 - \mathbf{x}_i : p dimensional vector
 - n : number of data points

Model: Linear Regression VS. Logistic Regression

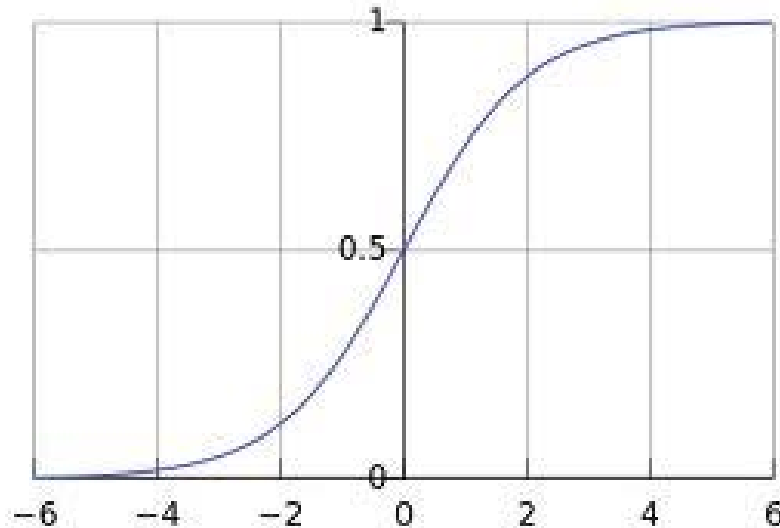
- Linear Regression (prediction)
 - Y : *continuous value* $(-\infty, +\infty)$
 - $y = \mathbf{x}^T \boldsymbol{\beta} = \beta_0 + x_1\beta_1 + x_2\beta_2 + \dots + x_p\beta_p$
 - $y|\mathbf{x}, \boldsymbol{\beta} \sim N(\mathbf{x}^T \boldsymbol{\beta}, \sigma^2)$
- Logistic Regression (classification)
 - Y : *discrete value from m classes*
 - $P(Y = j|\mathbf{x}, \boldsymbol{\beta}) \in [0,1]$ and $\sum_j P(Y = j|\mathbf{x}, \boldsymbol{\beta}) = 1$

Note: add additional constant feature 1 to original x to accommodate bias term

Logistic Function

- Logistic Function / sigmoid function:

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



Note: $\sigma'(x) = \sigma(x)(1 - \sigma(x))$

Modeling Probabilities of Two Classes

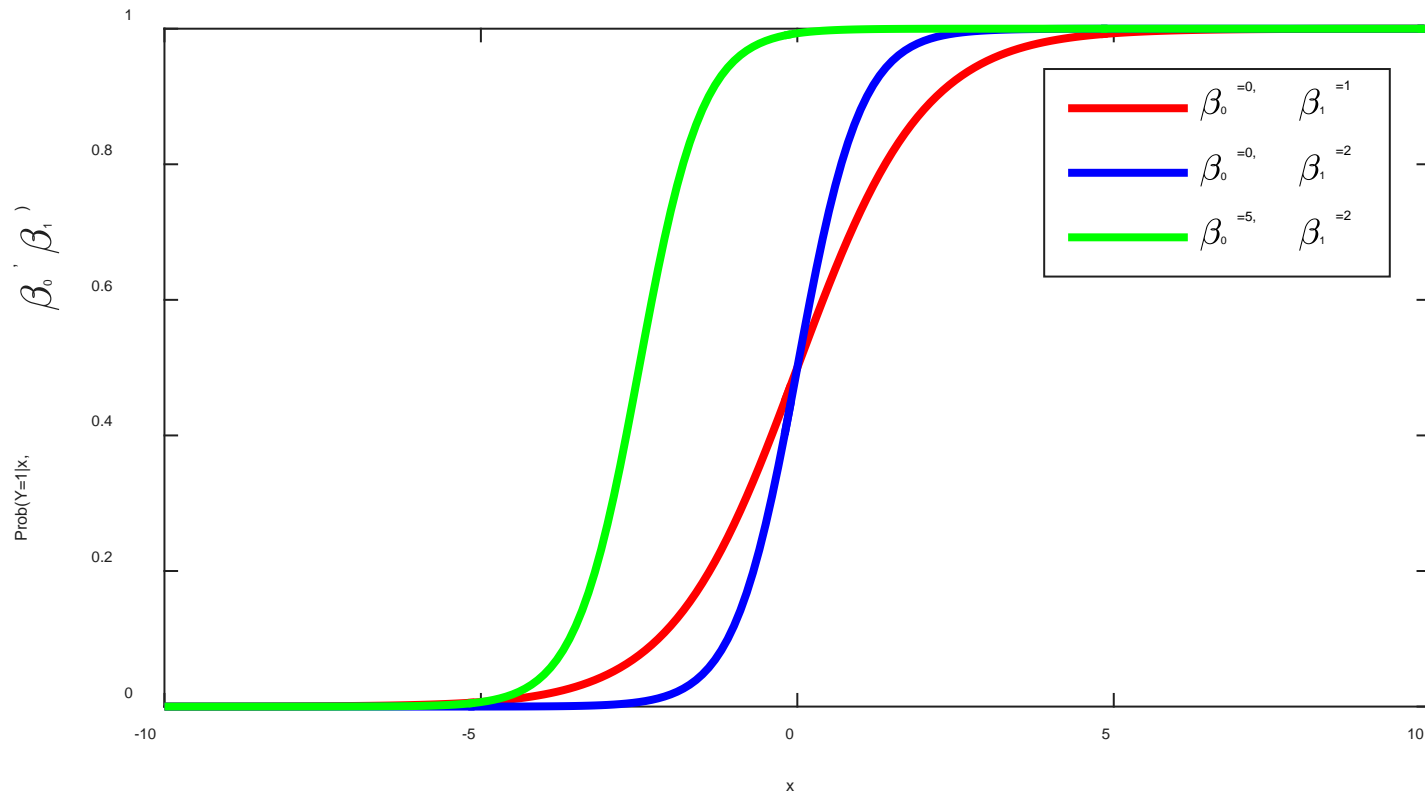
- $P(Y = 1|\mathbf{x}, \beta) = \sigma(\mathbf{x}^T \beta) = \frac{1}{1+\exp\{-\mathbf{x}^T \beta\}} = \frac{\exp\{\mathbf{x}^T \beta\}}{1+\exp\{\mathbf{x}^T \beta\}}$
- $P(Y = 0|\mathbf{x}, \beta) = 1 - \sigma(\mathbf{x}^T \beta) = \frac{\exp\{-\mathbf{x}^T \beta\}}{1+\exp\{-\mathbf{x}^T \beta\}} = \frac{1}{1+\exp\{\mathbf{x}^T \beta\}}$

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

- In other words
 - $y|\mathbf{x}, \beta \sim \text{Bernoulli}(\sigma(\mathbf{x}^T \beta))$

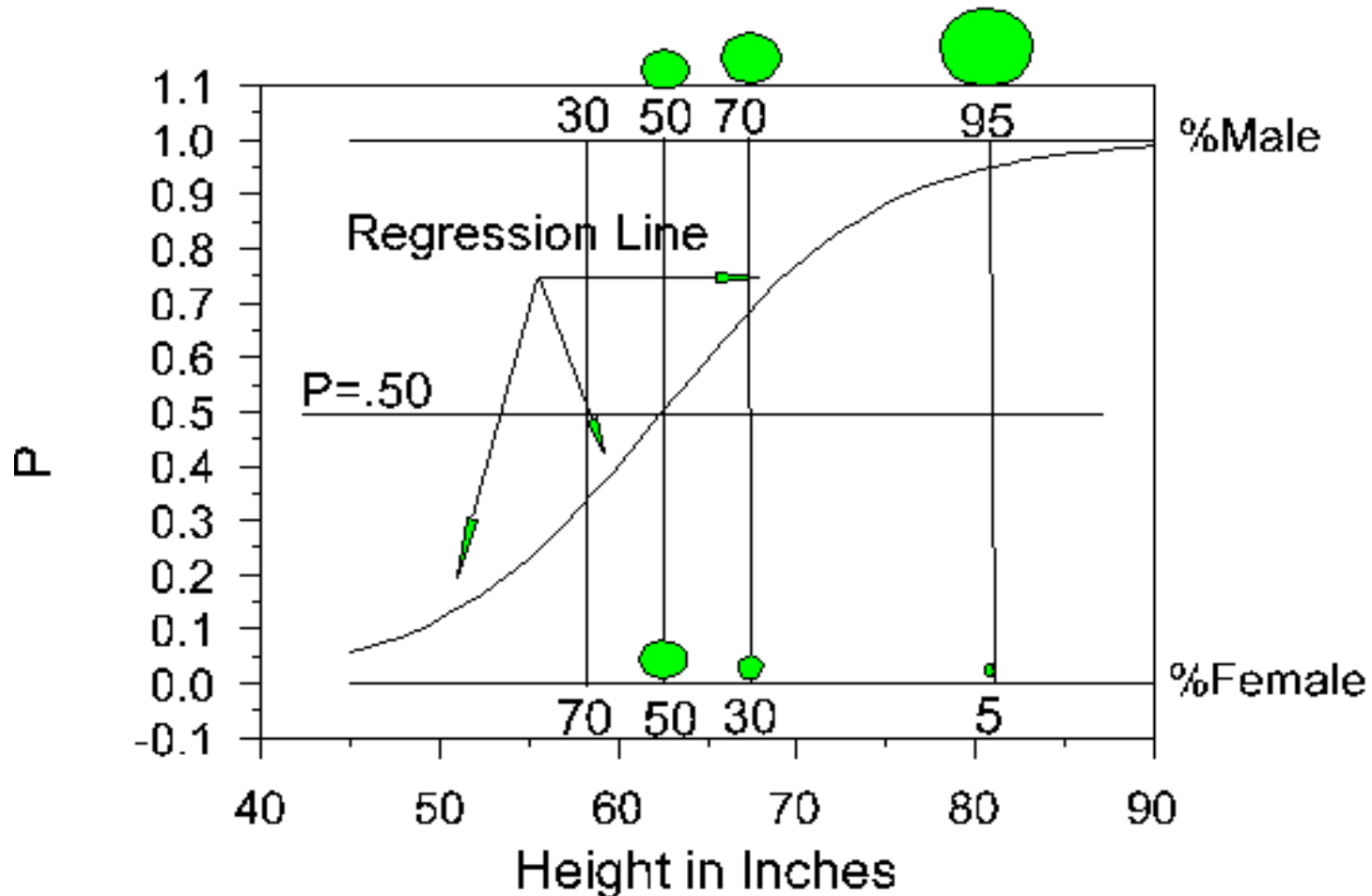
The 1-d Situation

- $P(Y = 1|x, \beta_0, \beta_1) = \sigma(\beta_1 x + \beta_0)$



Example

Regression of Sex on Height



Q: What is β_0 here?

Classification Assuming Parameters are Given: Inference

- *If* $P(Y = 1|\mathbf{x}, \beta) = \sigma(\mathbf{x}^T \beta) > 0.5$
 - Class 1
- Otherwise
 - Class 0

Parameter Estimation: Learning

- MLE estimation
 - Given a dataset D , with n data points
 - For a single data object with attributes \mathbf{x}_i , class label y_i
 - Let $p_i = p(y_i = 1 | \mathbf{x}_i, \beta)$, the prob. of i in class 1
 - The probability of observing y_i would be
 - If $y_i = 1$, then p_i
 - If $y_i = 0$, then $1 - p_i$
 - Combing the two cases: $p_i^{y_i}(1 - p_i)^{1-y_i}$

$$L = \prod_i p_i^{y_i}(1 - p_i)^{1-y_i} = \prod_i \left(\frac{\exp\{\mathbf{x}^T \beta\}}{1 + \exp\{\mathbf{x}^T \beta\}} \right)^{y_i} \left(\frac{1}{1 + \exp\{\mathbf{x}^T \beta\}} \right)^{1-y_i}$$

Optimization

- Equivalent to maximize log likelihood
- $\log L = \sum_i y_i \mathbf{x}_i^T \beta - \log(1 + \exp\{\mathbf{x}_i^T \beta\})$
- Gradient ascent update:

- $$\beta^{new} = \beta^{old} + \eta \frac{\partial \log L(\beta)}{\partial \beta}$$

Step size

- Newton-Raphson update

- $$\beta^{new} = \beta^{old} - \left(\frac{\partial^2 \log L(\beta)}{\partial \beta \partial \beta^T} \right)^{-1} \frac{\partial \log L(\beta)}{\partial \beta}$$

- where derivatives are evaluated at β^{old}

First Derivative

- It is a $(p+1)$ vector, with j th element as

$$\begin{aligned}\frac{\partial \log L(\beta)}{\partial \beta_j} &= \sum_{i=1}^N y_i x_{ij} - \sum_{i=1}^N \frac{x_{ij} e^{\beta^T \mathbf{x}_i}}{1 + e^{\beta^T \mathbf{x}_i}} \\ &= \sum_{i=1}^N y_i x_{ij} - \sum_{i=1}^N p_i(\beta) x_{ij} \\ &= \sum_{i=1}^N x_{ij} (y_i - p_i(\beta))\end{aligned}$$

$p_i(\beta) = \sigma(\beta^T \mathbf{x}_i)$

For $j = 0, 1, \dots, p$

Second Derivative

- It is a $(p+1)$ by $(p+1)$ matrix, Hessian Matrix, with j th row and n th column as

$$\begin{aligned}\frac{\partial \log L(\boldsymbol{\beta})}{\partial \beta_j \partial \beta_n} &= - \sum_{i=1}^N \frac{(1 + e^{\boldsymbol{\beta}^T \mathbf{x}_i}) e^{\boldsymbol{\beta}^T \mathbf{x}_i} x_{ij} x_{in} - (e^{\boldsymbol{\beta}^T \mathbf{x}_i})^2 x_{ij} x_{in}}{(1 + e^{\boldsymbol{\beta}^T \mathbf{x}_i})^2} \\ &= - \sum_{i=1}^N x_{ij} x_{in} p_i(\boldsymbol{\beta}) - \sum_{i=1}^N x_{ij} x_{in} (p_i(\boldsymbol{\beta}))^2 \\ &= - \sum_{i=1}^N x_{ij} x_{in} p_i(\boldsymbol{\beta}) (1 - p_i(\boldsymbol{\beta}))\end{aligned}$$

Matrix form:

$$\begin{aligned}\frac{\partial \log L(\boldsymbol{\beta})}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} &= - \sum_{i=1}^N \mathbf{x}_i p_i(\boldsymbol{\beta}) (1 - p_i(\boldsymbol{\beta})) \mathbf{x}_i^T \\ &= \mathbf{X}^T \begin{bmatrix} p_1(\boldsymbol{\beta})(1 - p_1(\boldsymbol{\beta})) & & \\ & \ddots & \\ & & p_N(\boldsymbol{\beta})(1 - p_N(\boldsymbol{\beta})) \end{bmatrix} \mathbf{X}\end{aligned}$$

where \mathbf{X} is the $N \times (p + 1)$ feature matrix. Note $\mathbf{X}^T = [x_1 \ x_2 \ \dots \ x_N]$.

What about Multiclass Classification?

- It is easy to handle under logistic regression, say M classes, using softmax function

- $$P(Y = j|x) = \frac{\exp\{x^T \beta_j\}}{1 + \sum_{m=1}^{M-1} \exp\{x^T \beta_m\}}, \text{ for } j = 1, \dots, M - 1$$

- $$P(Y = M|x) = \frac{1}{1 + \sum_{m=1}^{M-1} \exp\{x^T \beta_m\}}$$

Recall Linear Regression and Logistic Regression

- Linear Regression
 - $y|\mathbf{x}, \beta \sim N(\mathbf{x}^T \beta, \sigma^2)$
- Logistic Regression
 - $y|\mathbf{x}, \beta \sim \text{Bernoulli}(\sigma(\mathbf{x}^T \beta))$
- How about other distributions?
 - Yes, generalized linear models for exponential family

*Exponential Family

- Canonical Form

- $p(\mathbf{y}; \boldsymbol{\eta}) = b(\mathbf{y}) \exp(\boldsymbol{\eta}^T T(\mathbf{y}) - a(\boldsymbol{\eta}))$

- $\boldsymbol{\eta}$: natural parameter

- $T(\mathbf{y})$: sufficient statistic

- $a(\boldsymbol{\eta})$: log partition function for normalization

- $b(\mathbf{y})$: function that only dependent on \mathbf{y}

*Examples of Exponential Family

- Many:

$$p(y; \eta) = b(y) \exp(\eta^T T(y) - a(\eta))$$

- Gaussian, Bernoulli, Poisson, beta, Dirichlet, categorical, ...

- For Gaussian (not interested in σ)

$$\begin{aligned} p(y; \mu) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - \mu)^2\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) \cdot \exp\left(\mu y - \frac{1}{2}\mu^2\right) \end{aligned}$$

$$\begin{aligned} \eta &= \mu \\ T(y) &= y \\ a(\eta) &= \mu^2/2 \\ &= \eta^2/2 \\ b(y) &= (1/\sqrt{2\pi}) \exp(-y^2/2) \end{aligned}$$

- For Bernoulli

$$\begin{aligned} p(y; \phi) &= \phi^y (1 - \phi)^{1-y} \\ &= \exp(y \log \phi + (1 - y) \log(1 - \phi)) \\ &= \exp\left(\left(\log\left(\frac{\phi}{1 - \phi}\right)\right) y + \log(1 - \phi)\right) \end{aligned}$$


η

$$\begin{aligned} T(y) &= y \\ a(\eta) &= -\log(1 - \phi) \\ &= \log(1 + e^\eta) \\ b(y) &= 1 \end{aligned}$$

*Recipe of GLMs

- Determines a distribution for y
 - E.g., Gaussian, Bernoulli, Poisson
- Form the linear predictor for η
 - $\eta = \mathbf{x}^T \boldsymbol{\beta}$
- Determines a link function: $\mu = g^{-1}(\eta)$
 - Connects the linear predictor to the mean of the distribution
 - E.g., $\mu = \eta$ for Gaussian, $\mu = \sigma(\eta)$ for Bernoulli, $\mu = \exp(\eta)$ for Poisson

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Generative Models vs. Discriminative Models


- Generative model
 - *model joint probability $p(\mathbf{x}, y)$*
 - E.g., naïve Bayes
- Discriminative model
 - *model conditional probability $p(y|\mathbf{x})$*
 - E.g., logistic regression

Which One is Better?

- Consider $p(\mathbf{x}, y) = p(y|\mathbf{x}) \times p(\mathbf{x})$
 - Generative models require additional model of marginal distribution $p(\mathbf{x})$
 - Need more data to learn $p(\mathbf{x})$
 - Distribution assumption of $p(\mathbf{x})$ might be incorrect
 - In practice, discriminative models work very well

<https://ai.stanford.edu/~ang/papers/nips01-discriminativegenerative.pdf>

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Summary

- Probabilistic Models
 - I.I.D. assumption enables joint distribution of data as a product of probability of single data points
- Naïve Bayes
 - Assuming independence among features
- Logistic Regression
 - Assuming conditional distribution follows Bernoulli distribution
- Generative Models and Discriminative Models
 - Modeling joint distribution vs. conditional distribution

References

- <http://pages.cs.wisc.edu/~jerryzhu/cs769/nb.pdf>
- <http://cs229.stanford.edu/notes/cs229-notes1.pdf>
- <https://ai.stanford.edu/~ang/papers/nips01-discriminativegenerative.pdf>

More about Lagrangian

- Objective with equality constraints

$$\min_w f(w)$$

s. t.

$$h_i(w) = 0, \text{ for } i = 1, 2, \dots, l$$

- Lagrangian:

- $L(w, \alpha) = f(w) + \sum_i \alpha_i h_i(w)$

- α_i : Lagrangian multipliers

- Solution: setting the derivatives of Lagrangian to be 0

- $\frac{\partial L}{\partial w} = 0$ and $\frac{\partial L}{\partial \alpha_i} = 0$ for every i

Generalized Lagrangian

- Objective with both equality and inequality constraints

$$\min_w f(w)$$

s. t.

$$h_i(w) = 0, \text{ for } i = 1, 2, \dots, l$$

$$g_j(w) \leq 0, \text{ for } j = 1, 2, \dots, k$$

- Lagrangian

- $L(w, \alpha, \beta) = f(w) + \sum_i \alpha_i h_i(w) + \sum_j \beta_j g_j(w)$

- α_i : Lagrangian multipliers

- $\beta_j \geq 0$: Lagrangian multipliers

Why It Works

- Consider function

$$\theta_p(w) = \max_{\alpha, \beta: \beta_j \geq 0} L(w, \alpha, \beta)$$

- $\theta_p(w) = \begin{cases} f(w), & \text{if } w \text{ satisfies all constraints} \\ \infty, & \text{if } w \text{ doesn't satisfy constraints} \end{cases}$
- Therefore, minimize $f(w)$ with constraints is equivalent to minimize $\theta_p(w)$

Lagrange Duality

- The primal problem

$$p^* = \min_w \max_{\alpha, \beta: \beta_j \geq 0} L(w, \alpha, \beta)$$

- The dual problem

$$d^* = \max_{\alpha, \beta: \beta_j \geq 0} \min_w L(w, \alpha, \beta)$$

- According to max-min inequality

$$p^* \geq d^*$$

- When does equation hold?

Primal = Dual

- $p^* = d^*$, under some proper condition (Slater conditions)
 - f, g_j convex, h_i affine
 - Exists w , such that all $g_j(w) < 0$
- (w^*, α^*, β^*) need to satisfy KKT conditions
 - $\frac{\partial L}{\partial w} = 0$
 - $\beta_j g_j(w) = 0$
 - $h_i(w) = 0, g_j(w) \leq 0, \beta_j \geq 0$

https://cs.stanford.edu/people/davidknowles/lagrangian_duality.pdf