

# CS247: ADVANCED DATA MINING

## Graph and Network: Random Walk

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# Methods to Learn

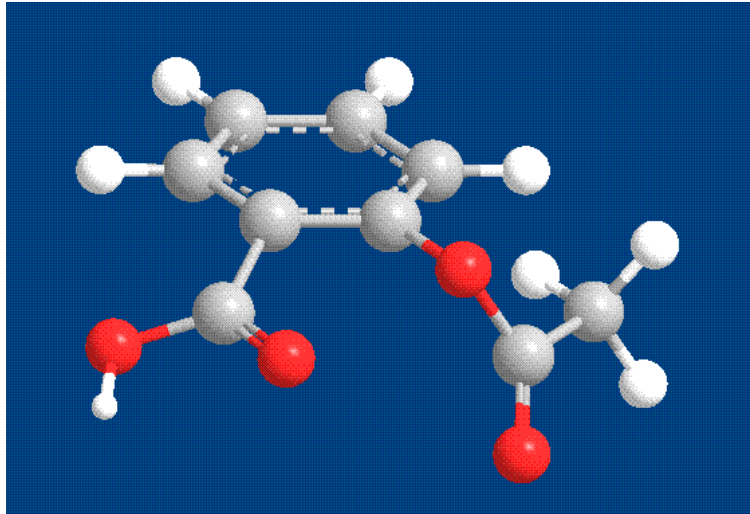
	Vector Data	Text Data	Graph & Network	Recommender Systems
Classification	Naïve Bayes; Logistic Regression; NN		Label Propagation	
Clustering	K-means; Mixture Models	PLSA; LDA	Spectral Clustering	Matrix Factorization
Prediction	NN			Collaborative Filtering; Factorization machine; Hybrid CF; Recommendation with graph regularization
Ranking			PageRank	
Similarity Search			P-PageRank	
Representation Learning		Word embedding	Network embedding	Deep collaborative learning

# Random Walk on Graphs

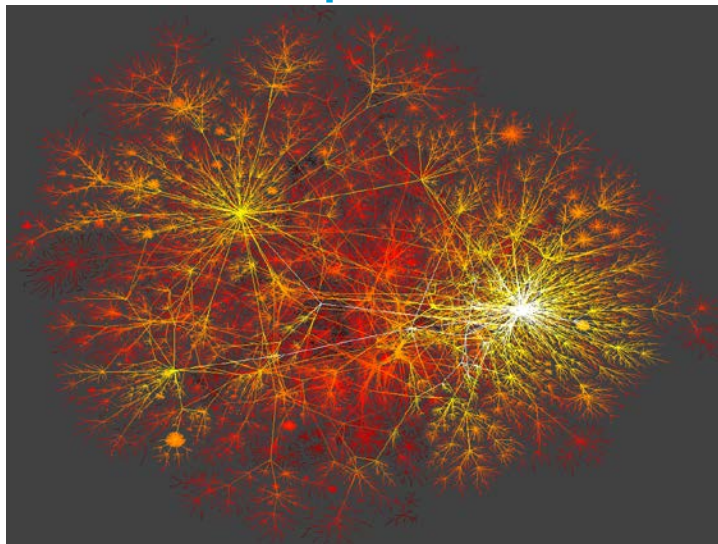
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- Introduction to Graph/Network Data 
- PageRank
- Personalized PageRank
- Summary

# Graph, Graph, Everywhere



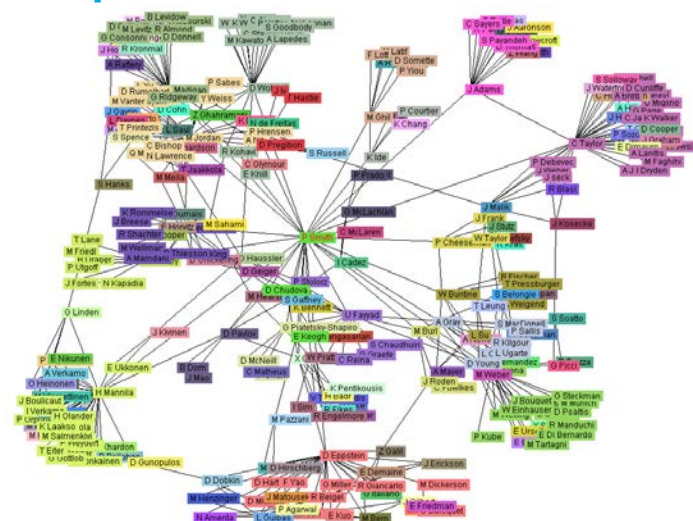
Aspirin



Internet



Yeast protein interaction network



Co-author network

from H. Jeong et al Nature 411, 41 (2001)

# Why Graph Mining?

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- Graphs are ubiquitous
  - Chemical compounds (Cheminformatics)
  - Protein structures, biological pathways/networks (Bioinformatics)
  - Program control flow, traffic flow, and workflow analysis
  - XML databases, Web, and social network analysis
- Graph is a general model
  - Trees, lattices, sequences, and items are degenerated graphs
- Diversity of graphs
  - Directed vs. undirected, labeled vs. unlabeled (edges & vertices), weighted vs. unweighted, homogeneous vs. heterogeneous
- Complexity of algorithms: many problems are of high complexity

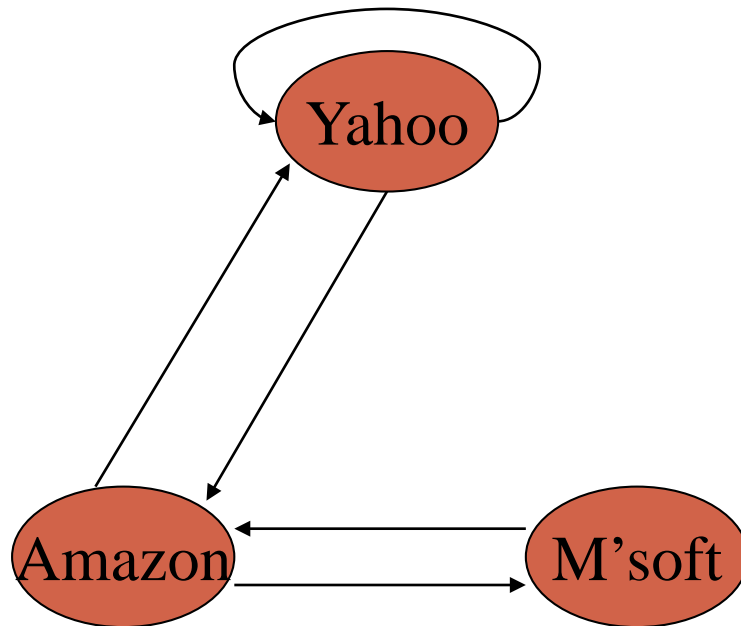
# Representation of a Graph

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- $G = \langle V, E \rangle$ 
  - $V = \{u_1, \dots, u_n\}$ : node set
  - $E \subseteq V \times V$ : edge set
- Adjacency matrix
  - $A = \{a_{ij}\}, i, j = 1, \dots, N$ 
    - $a_{ij} = 1, \text{ if } \langle u_i, u_j \rangle \in E$
    - $a_{ij} = 0, \text{ if } \langle u_i, u_j \rangle \notin E$
  - Undirected graph vs. Directed graph
    - $A = A^T$  vs.  $A \neq A^T$
  - Weighted graph
    - Use  $W$  instead of  $A$ , where  $w_{ij}$  represents the weight of edge  $\langle u_i, u_j \rangle$

# Example

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	y	a	m
y	1	1	0
a	1	0	1
m	0	1	0

Adjacency matrix A

# Random Walk on Graphs

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# The History of PageRank

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- PageRank was developed by Larry Page (hence the name *Page-Rank*) and Sergey Brin.
- It is first as part of a research project about a new kind of search engine. That project started in 1995 and led to a functional prototype in 1998.
- Shortly after, Page and Brin founded Google.

# Ranking web pages

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- Web pages are not equally “important”
  - [www.cnn.com](http://www.cnn.com) vs. a personal webpage
- Inlinks as votes
  - The more inlinks, the more important
- Are all inlinks equal?
  - Higher ranked inlink should play a more important role
  - Recursive question!

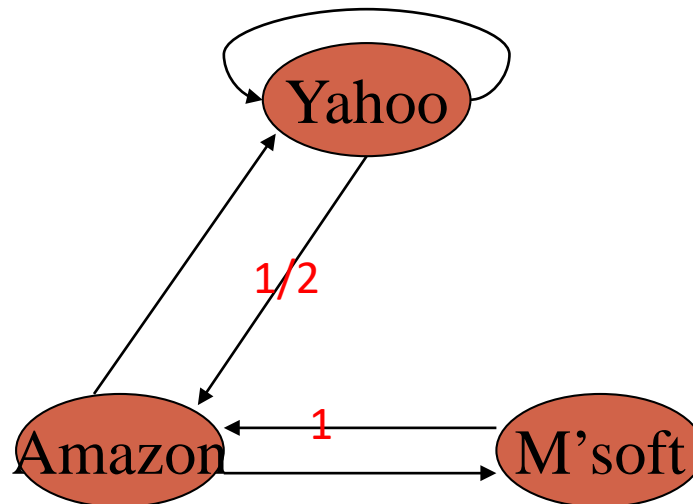
# Simple recursive formulation

- Each link's vote is proportional to the **importance** of its source page
- If page **P** with importance **x** has **n** outlinks, each link gets **x/n** votes
- Page **P**'s own importance is the sum of the votes on its inlinks

$$y = y/2 + a/2$$

$$a = y/2 + m$$

$$m = a/2$$



# Matrix Formulation

- N by N Matrix **M**
- Suppose page j has n outlinks

	y	a	m	
y	1	1	0	$\frac{1}{2}$ $\frac{1}{2}$ 0
a	1	0	1	$\frac{1}{2}$ 0 1
m	0	1	0	0 $\frac{1}{2}$ 0



- If  $j \rightarrow i$ , then  $M_{ij} = 1/n$ ; Else  $M_{ij} = 0$
- Or  $M = (D^{-1}A)^T$ , where **D** is a diagonal matrix and  $D_{ii} = \sum_j A_{ij}$
- **M** is a **column stochastic matrix**
  - Columns sum to 1
- Suppose **r** is a vector with one entry per web page
  - $r_i$  is the importance score of page i
  - Call it the **rank vector**
  - $|\mathbf{r}| = 1$  (i.e.,  $r_1 + r_2 + \dots + r_N = 1$ )

# Eigenvector formulation

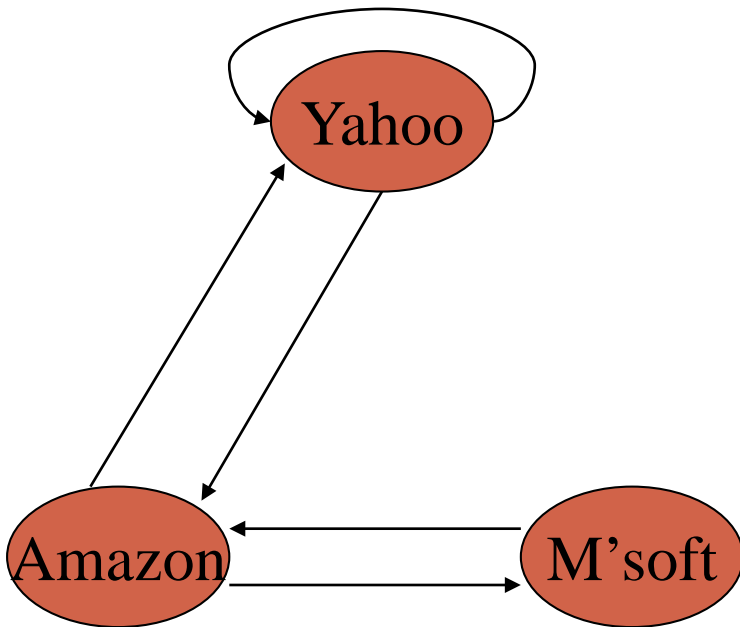
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- The flow equations can be written

$$r = Mr$$

- So the rank vector is an eigenvector of the stochastic web matrix
  - In fact, it's first or principal eigenvector, with corresponding eigenvalue 1

# Example



$$y = y/2 + a/2$$

$$a = y/2 + m$$

$$m = a/2$$

	y	a	m
y	1/2	1/2	0
a	1/2	0	1
m	0	1/2	0

$$\mathbf{r} = \mathbf{M} * \mathbf{r}$$

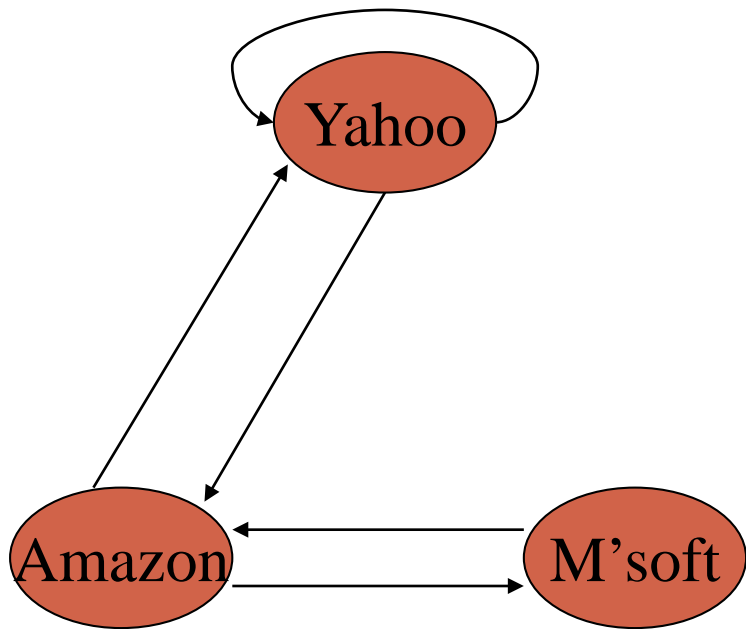
y	1/2	1/2	0	y
a	1/2	0	1	a
m	0	1/2	0	m

# Power Iteration method

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- Simple iterative scheme
- Suppose there are  $N$  web pages
  - Initialize:  $\mathbf{r}^0 = [1/N, \dots, 1/N]^T$
  - Iterate:  $\mathbf{r}^{k+1} = \mathbf{M}\mathbf{r}^k$
  - Stop when  $\|\mathbf{r}^{k+1} - \mathbf{r}^k\|_1 < \varepsilon$ 
    - $\|\mathbf{x}\|_1 = \sum_{1 \leq i \leq N} |x_i|$  is the  $L_1$  norm
    - Can use any other vector norm e.g., Euclidean

# Power Iteration Example



	y	a	m
y	1/2	1/2	0
a	1/2	0	1
m	0	1/2	0

y	=	1/3	1/3	5/12	3/8		2/5
a		1/3	1/2	1/3	11/24	...	2/5
m		1/3	1/6	1/4	1/6		1/5
		$r_0$	$r_1$	$r_2$	$r_3$	...	$r^*$

# Random Walk Interpretation

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- Imagine a **random web surfer**
  - At any time  $t$ , surfer is on some page  $P$
  - At time  $t+1$ , the surfer follows an outlink from  $P$  uniformly at random
  - Ends up on some page  $Q$  linked from  $P$
  - Process repeats indefinitely
- Let  $\mathbf{p}(t)$  be a vector whose  $i^{\text{th}}$  component is the probability that the surfer is at page  $i$  at time  $t$ 
  - $\mathbf{p}(t)$  is a probability distribution on pages

# The stationary distribution

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- Where is the surfer at time  $t+1$ ?
  - Follows a link uniformly at random
  - $\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)$
- Suppose the random walk reaches a state such that  $\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t) = \mathbf{p}(t)$ 
  - Then  $\mathbf{p}(t)$  is called a **stationary distribution** for the random walk
- Our rank vector  $\mathbf{r}$  satisfies  $\mathbf{r} = \mathbf{M}\mathbf{r}$ 
  - So it is a stationary distribution for the random surfer

# Existence and Uniqueness

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A central result from the theory of random walks (aka Markov processes):

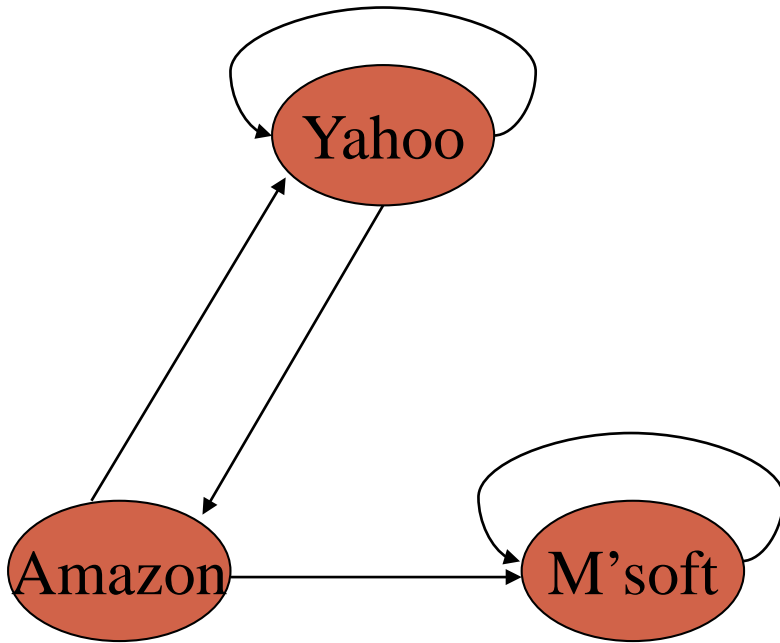
For graphs that satisfy certain conditions, the stationary distribution is unique and eventually will be reached no matter what the initial probability distribution at time  $t = 0$ .

# Spider traps

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- A group of pages is a **spider trap** if there are no links from within the group to outside the group
  - Random surfer gets trapped!

# Microsoft becomes a spider trap



	y	a	m
y	1/2	1/2	0
a	1/2	0	0
m	0	1/2	1

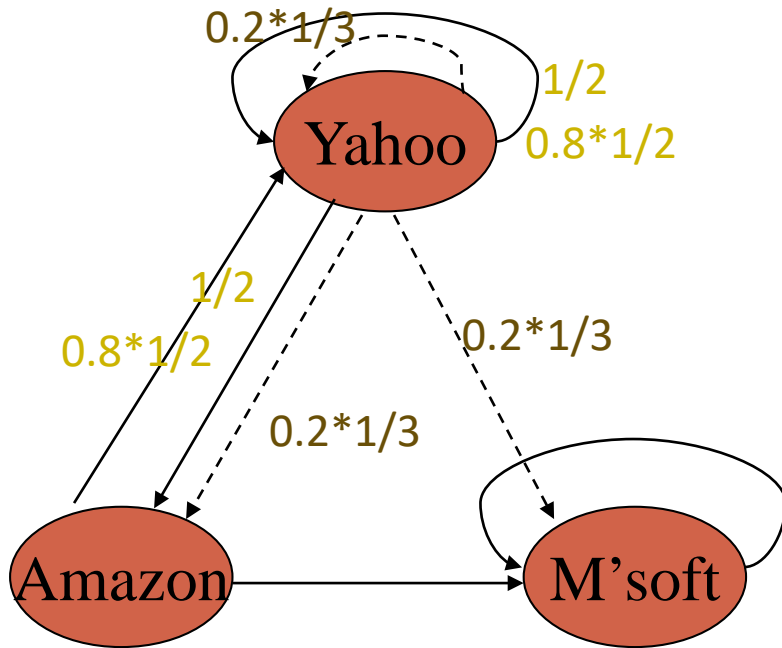
y	=	1/3	1/3	1/4	5/24		0
a		1/3	1/6	1/6	1/8	...	0
m		1/3	1/2	7/12	2/3		1

# Random teleports

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- The Google solution for spider traps
- At each time step, the random surfer has two options:
  - With probability  $\beta$ , follow a link at random
  - With probability  $1-\beta$ , jump to some page uniformly at random
  - Common values for  $\beta$  are in the range 0.8 to 0.9
- Surfer will teleport out of spider trap within a few time steps

# Random teleports ( $\beta = 0.8$ )



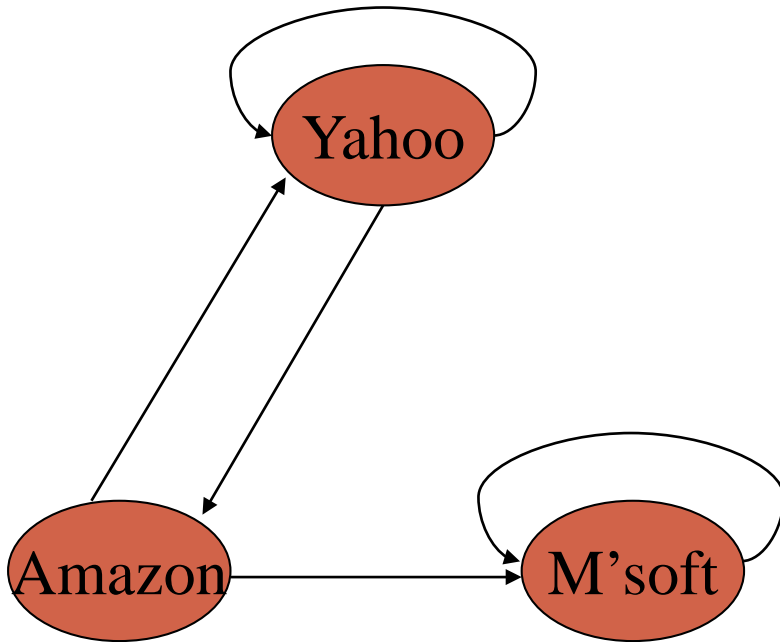
$$\begin{array}{c} y \\ a \\ m \end{array} \begin{array}{c} y \\ 1/2 \\ 1/2 \\ 0 \end{array} \quad 0.8 * \begin{array}{c} y \\ 1/2 \\ 1/2 \\ 0 \end{array} \quad + \quad 0.2 * \begin{array}{c} y \\ 1/3 \\ 1/3 \\ 1/3 \end{array}$$

$$0.8 \begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{array} \quad + \quad 0.2 \begin{array}{ccc} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{array}$$

-----> : teleport links from "Yahoo"

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{array}{ccc} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{array}$$

# Random teleports ( $\beta = 0.8$ )



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{matrix} y \\ a \\ m \end{matrix} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{matrix} y \\ a \\ m \end{matrix} = \begin{bmatrix} 0.333 \\ 0.333 \\ 0.333 \end{bmatrix} \begin{bmatrix} 0.333 \\ 0.200 \\ 0.467 \end{bmatrix} \begin{bmatrix} 0.280 \\ 0.200 \\ 0.520 \end{bmatrix} \begin{bmatrix} 0.259 \\ 0.179 \\ 0.563 \end{bmatrix} \dots \begin{bmatrix} 7/33 \\ 5/33 \\ 21/33 \end{bmatrix}$$

# Matrix formulation

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- Suppose there are  $N$  pages
  - Consider a page  $j$ , with set of outlinks  $O(j)$
  - We have  $M_{ij} = 1/|O(j)|$  when  $j \rightarrow i$  and  $M_{ij} = 0$  otherwise
  - The random teleport is equivalent to
    - adding a **teleport link** from  $j$  to every other page with probability  $(1-\beta)/N$
    - reducing the probability of following each outlink from  $1/|O(j)|$  to  $\beta/|O(j)|$
    - Equivalent: tax each page a fraction  $(1-\beta)$  of its score and redistribute evenly

# PageRank

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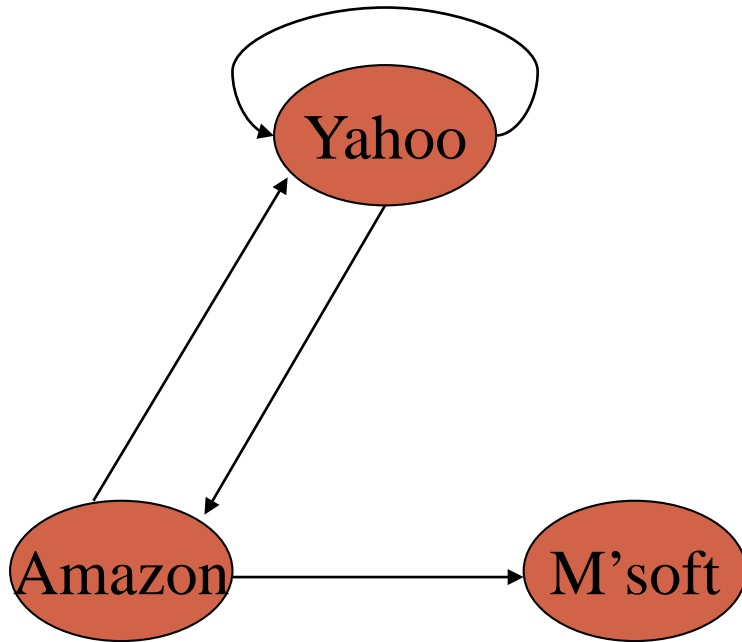
- Construct the N-by-N matrix **B** as follows
  - $B_{ij} = \beta M_{ij} + (1-\beta)/N$
- Verify that **B** is a stochastic matrix
- The **page rank vector** **r** is the principal eigenvector of this matrix
  - satisfying  $\mathbf{r} = \mathbf{B}\mathbf{r}$
- Equivalently, **r** is the stationary distribution of the random walk with teleports

# Dead ends

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- Pages with no outlinks are “dead ends” for the random surfer
  - Nowhere to go on next step

# Microsoft becomes a dead end



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$$

$$+ 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{matrix} y \\ a \\ m \end{matrix} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 1/15 \end{bmatrix}$$

$$\begin{matrix} y \\ a \\ m \end{matrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 0.2 \\ 1/3 & 0.2 \end{bmatrix} \begin{matrix} 0 \\ \dots \\ 0 \end{matrix}$$

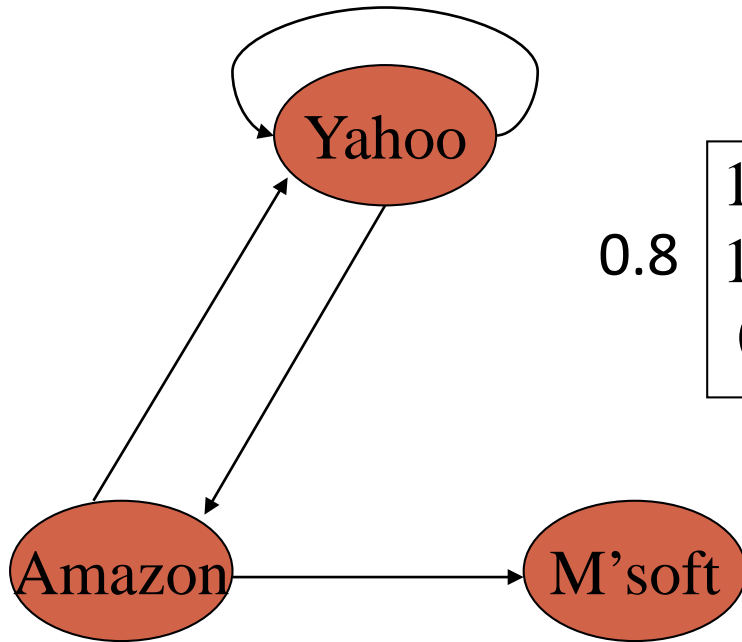
↓  
Non-stochastic!

# Dealing with dead-ends

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- Method 1: Teleport
  - Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly
- Method 2: Prune and propagate
  - Preprocess the graph to eliminate dead-ends
    - Might require multiple passes
  - Compute pagerank on reduced graph
  - Approximate values for deadends by propagating values from reduced graph

# Dealing dead end: teleport

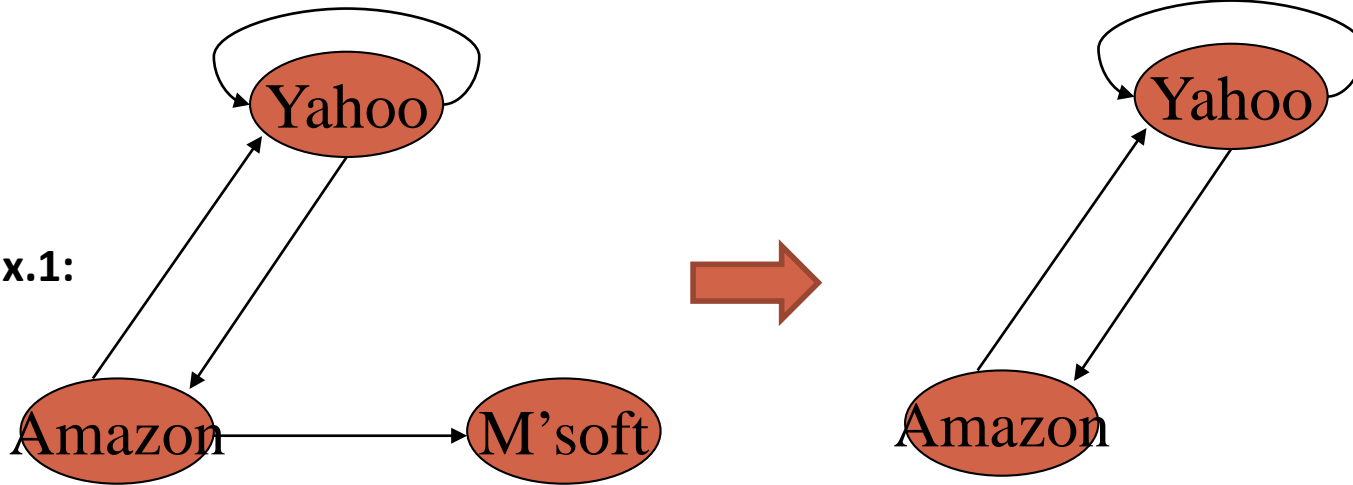


$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} + \begin{bmatrix} 0.2*1/3 & 0.2*1/3 & 1*1/3 \\ 0.2*1/3 & 0.2*1/3 & 1*1/3 \\ 0.2*1/3 & 0.2*1/3 & 1*1/3 \end{bmatrix}$$

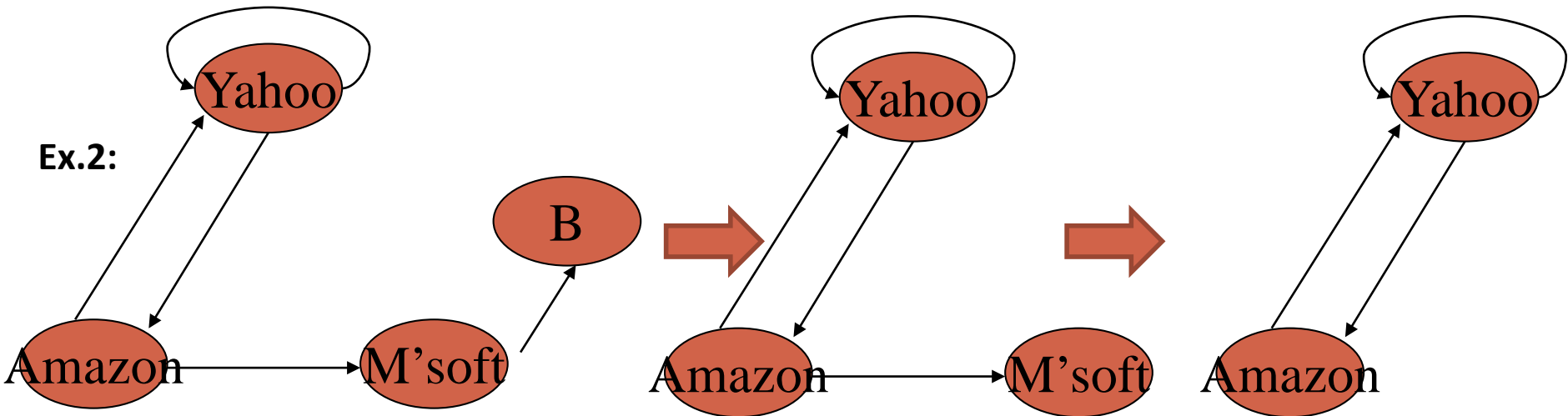
$$\begin{matrix} y \\ a \\ m \end{matrix} \begin{bmatrix} 7/15 & 7/15 & 1/3 \\ 7/15 & 1/15 & 1/3 \\ 1/15 & 7/15 & 1/3 \end{bmatrix}$$

# Dealing dead end: reduce graph

Ex.1:



Ex.2:



# Computing PageRank

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- Key step is matrix-vector multiplication
  - $\mathbf{r}^{\text{new}} = \mathbf{B}\mathbf{r}^{\text{old}}$
- Easy if we have enough main memory to hold  $\mathbf{B}$ ,  $\mathbf{r}^{\text{old}}$ ,  $\mathbf{r}^{\text{new}}$
- Say  $N = 1$  billion pages
  - We need 4 bytes for each entry (say)
  - 2 billion entries for vectors, approx 8GB
  - Matrix  $\mathbf{B}$  has  $N^2$  entries
    - $10^{18}$  is a large number!

# Rearranging the equation

---

$\mathbf{r} = \mathbf{B}\mathbf{r}$ , where

$$B_{ij} = \beta M_{ij} + (1-\beta)/N$$

$$r_i = \sum_{1 \leq j \leq N} B_{ij} r_j$$

$$r_i = \sum_{1 \leq j \leq N} [\beta M_{ij} + (1-\beta)/N] r_j$$

$$= \beta \sum_{1 \leq j \leq N} M_{ij} r_j + (1-\beta)/N \sum_{1 \leq j \leq N} r_j$$

$$= \beta \sum_{1 \leq j \leq N} M_{ij} r_j + (1-\beta)/N, \text{ since } |\mathbf{r}| = 1$$

$$\mathbf{r} = \beta \mathbf{M}\mathbf{r} + [(1-\beta)/N]_N$$

where  $[x]_N$  is an N-vector with all entries x

# Sparse matrix formulation

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- We can rearrange the page rank equation:
  - $\mathbf{r} = \beta \mathbf{M} \mathbf{r} + [(1-\beta)/N]_N$
  - $[(1-\beta)/N]_N$  is an  $N$ -vector with all entries  $(1-\beta)/N$
- $\mathbf{M}$  is a sparse matrix!
  - 10 links per node, approx  $10N$  entries
- So in each iteration, we need to:
  - Compute  $\mathbf{r}^{\text{new}} = \beta \mathbf{M} \mathbf{r}^{\text{old}}$
  - Add a constant value  $(1-\beta)/N$  to each entry in  $\mathbf{r}^{\text{new}}$

# Sparse matrix encoding

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- Encode sparse matrix using only nonzero entries
  - Space proportional roughly to number of links
  - say  $10N$ , or  $4 * 10 * 1$  billion = 40GB
  - still won't fit in memory, but will fit on disk

source node	degree	destination nodes
0	3	1, 5, 7
1	5	17, 64, 113, 117, 245
2	2	13, 23

# Basic Algorithm

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- Assume we have enough RAM to fit  $\mathbf{r}^{\text{new}}$ , plus some working memory
  - Store  $\mathbf{r}^{\text{old}}$  and matrix  $\mathbf{M}$  on disk

## Basic Algorithm:

- Initialize:  $\mathbf{r}^{\text{old}} = [1/N]_N$
- Iterate:
  - **Update:** Perform a sequential scan of  $\mathbf{M}$  and  $\mathbf{r}^{\text{old}}$  to update  $\mathbf{r}^{\text{new}}$
  - Write out  $\mathbf{r}^{\text{new}}$  to disk as  $\mathbf{r}^{\text{old}}$  for next iteration
  - Every few iterations, compute  $|\mathbf{r}^{\text{new}} - \mathbf{r}^{\text{old}}|$  and stop if it is below threshold
    - Need to read in both vectors into memory

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# Personalized PageRank

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- Query-dependent Ranking
  - For a query webpage  $u$ , which webpages are most important to  $u$ ?
    - We need a measure  $s(u,v)$
  - The relative important webpages to different queries would be different

# Calculation of P-PageRank

- Recall PageRank calculation:

- $\mathbf{r} = \beta \mathbf{M} \mathbf{r} + [(1-\beta)/N] \mathbf{1}_N$  or

- $\mathbf{r} = \beta \mathbf{M} \mathbf{r} + (1-\beta) \mathbf{r}_0$ , where  $\mathbf{r}_0 = \begin{pmatrix} 1/N \\ 1/N \\ \dots \\ 1/N \end{pmatrix}$


- For P-PageRank,  $s(u,v) = \mathbf{r}_u(v)$ , where  $\mathbf{r}_u = \beta \mathbf{M} \mathbf{r}_u + (1-\beta) \mathbf{r}_0$

by replacing  $\mathbf{r}_0$  with  $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}$  ← uth webpage

- Teleport to webpage  $u$

# Random Walk on Graphs

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# Summary

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- Random walk on graph
  - PageRank for ranking
  - P-PageRank for similarity search