University of California, Los Angeles
CS 289A Communication Complexity
Instructor: Alexander Sherstov

Date assigned: March 12, 2012
Date due: March 21, 2012

Problem Set II

1. We proved in class that DISJ_n has unbounded-error communication complexity $O(\log n)$. Prove that this bound is asymptotically tight. You must solve this problem from first principles, without appeal to the techniques of Lecture 15 such as Forster's method.

2. Determine the unbounded-error communication complexity of EQ_n, up to a multiplicative constant.

3. A *Euclidean embedding* of $A \in \{-1,+1\}^{n\times m}$ is a system of vectors $u_1,\dots,u_n,v_1,\dots,v_m \in \mathbb{R}^k$ (for finite but arbitrarily large k) such that $A_{ij} = \operatorname{sgn}\langle u_i, v_j\rangle$ for all i,j. The *margin* of this Euclidean embedding is defined as $\min_{i,j} |\langle u_i/\|u_i\|, v_j/\|v_j\|\rangle|$. The *margin of A*, denoted $m(A)$, is the supremum of the margin over all Euclidean embeddings of A. Prove that $c \leqslant \operatorname{disc}(A)/m(A) \leqslant C$ for some absolute constants $C \geqslant c > 0$ and all $A \in \{-1,+1\}^{n\times m}$.

4. A village has n residents. Every day at noon, they all meet at the main square to discuss daily matters. An evil spirit visits the village one night and marks the ears of k of the villagers with indelible ink, $k \geqslant 1$. Later that day, at noon, the evil spirit comes to the village meeting and announces that at least one villager has a marked ear. The evil spirit never visits the village again. If (and only if) a villager is able to logically deduce that he has a marked ear, he will leave the village the same day, never to be seen again. Ear marks being a taboo subject in town, the villagers never discuss it in any way. Moreover, a villager with a marked ear can never see his own mark in a mirror or otherwise. What will be the population count in the village $n+1$ days after the evil spirit's visit?

5. Prove that the containment $\Sigma_1^{cc} \cup \Pi_1^{cc} \subset \Sigma_2^{cc} \cup \Pi_2^{cc}$ is strict.

6. Define $E(n,d) = \min_p \sum_{x\in\{0,1\}^n} |p(x_1,x_2,\dots,x_n) - x_1x_2\cdots x_n|$, where the minimum is over all polynomials p of degree at most d. In words, $E(n,d)$ is the least ℓ_1 error in an approximation of the AND function by a polynomial of degree at most d. Determine $E(n,d)$ for $d = 0,1,2,\dots,n$.

7. Let $N_1,N_2,\dots,N_k$ be norms on $\mathbb{R}^n$. Consider the norm N given by $N = \max\{N_1,N_2,\dots,N_k\}$. Prove that $B_N = \bigcap B_{N_i}$ and $B_{N^*} = \operatorname{conv}(\bigcup B_{N_i^*})$.

8. Complete the proof of Forster's theorem by showing that $\operatorname{range}(\sum_{x\in X} xx^\mathsf{T}) = \operatorname{span} X$ for any finite $X \subset \mathbb{R}^n$.

9. A *Boolean formula* in Boolean variables $z_1,\dots,z_n$ is a fully parenthesized Boolean expression with literals $z_1,\neg z_1,\dots,z_n,\neg z_n$ and binary operators $\wedge$ and $\vee$. Formally, a Boolean formula is recursively defined as follows: each of z_i and $\neg z_i$ is a Boolean formula; if F and G are Boolean formulas, then so are $(F)\wedge(G)$ and $(F)\vee(G)$. Let $F(z_1,\dots,z_n)$ be a Boolean formula in which every variable occurs exactly once. Determine the deterministic communication complexity of $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ given by $f(x,y) = F(x_1\oplus y_1,\dots,x_n\oplus y_n)$.

10. For a random $A \in \{-1,+1\}^{n\times n}$, prove that $\mathbf{P}[|\|A\| - \mathbf{E}\,\|A\|| \geqslant t] \leqslant \exp(-\Omega(t^2))$.